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PAPERS

STRUCTURAL BEAMS IN TORSION

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and Bruce G. Johnston,[2] Jun. Am. Soc. C. E.

Synopsis

Results of a study of the torsional properties of standard structural steel beams are offered for discussion in this paper. The purpose of the investigation was to furnish a reliable basis for the design of structural members subjected to torsional loads. The relation between torque and stress on the one hand, and between torque and twist on the other, for any piece subjected to torsion involves a constant the value of which is a function of the material and the shape of the cross-section. An accurate method is given for the evaluation of this torsion constant, K, for standard **H**-sections and **I**-sections, taking full account of all factors involved. This has been made possible by applying the "membrane analogy" to about sixty sections of widely varying flange, web, and fillet proportions.

The investigation included a study of the effect of end fixity in torsional design, and shows how it may be obtained effectively. The proposed formulas are applied to practical design problems, and are checked by torsional tests on structural steel sections ranging in size from a 3-in. I-beam weighing 7.5 lb per ft, to a 12 by 12-in. beam weighing 190 lb per ft.

Historical Foreword

The problem of pure torsion as applied to non-circular sections was first treated correctly by Saint Venant (1)[3] in 1855, and his general solution is applicable to any cross-section. In 1903, Prandtl (2) showed that if a thin membrane were stretched across a hole having the shape of the cross-section in question and distorted slightly, the equation of its surface had the same form as the general differential equation involved in the torsion problem.

Note.—Discussion on this paper will be closed in August, 1935, *Proceedings*.

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[3] The numbers in parentheses refer to references given in Appendix I.

Prandtl showed that by measuring the volume and slopes of the displaced membrane a direct measurement of the torsional rigidity and stress was obtainable. Prandtl's analogy, with a thin soap film as a membrane, was used in several torsion investigations, first in England by Griffith and Taylor (3) who studied the torsional strength of aeroplane sections in 1917, and, later, in the United States by Trayer and March (4), who, in 1930, made similar studies for the same purpose.

Important contributions to the torsion problem have been made by Timoshenko (5). He has shortened the pure torsional theory by slight modifications of Saint Venant's equations and by mathematical application of the principles of the membrane analogy. He was also among the first to consider the effect produced by preventing the warping of a cross-section. This problem has had the attention of numerous investigators in connection with problems of elastic stability and buckling during bending. Sonntag (7) treated the theoretical aspects of this problem in an article published in 1929.

Introduction

The investigation reported herein was undertaken as a study of all available information on the subject, both theoretical and experimental, supplemented by a considerable number of actual torsion tests of structural steel beams and soap film experiments on various cross-sectional shapes. The writers first considered testing beam sections 3 ft in length, welded to thick plates at the ends. A study of the problem, however, showed that such beams would be several times stronger than if they were tested free-ended, and that, unless the exact percentage of end fixity was known, it would not be possible to draw definite conclusions from such tests.

In order to study the effect of end fixity directly, tests were first made on eight sections of a 3-in. I-beam (7.5 lb per ft), varying from 3 in. to 4 ft 6 in. in length and cut from the same rolled section. The ends of each piece were welded to plates 1 in. thick, and the specimens were tested in a standard 26 000 in-lb torsion machine.

The results of these tests pointed the way to a revised general program, and a torsion testing rig capable of applying torsional load up to 750 000 in-lb was designed. Provision was made for testing beams either fixed or free at the ends, and with lengths of 1 ft 6 in., 3 ft, or 6 ft. Nineteen different tests were made, twelve on beams with the ends fixed by welding the side and end plates to form a box section at the ends, and seven with ends free. The beams ranged in sizes from the 3-in. I-beam, weighing 7.5 lb per ft, to a 12 by 12-in. beam weighing 190 lb per ft. Tensile and shearing properties of the material in each type of beam were obtained by standard tensile tests, round bar torsion tests, and slotted plate shear tests. Soap film experiments on fifty-seven differently proportioned sections were made for the determination of the torsional rigidity.

This paper contains the final summary of all phases of the investigation. Use has been freely made of the findings of previous investigators, for which acknowledgment is made at appropriate points.

Notation.—The symbols in this paper are introduced in the text as they occur, and are summarized for reference in Appendix II.

THE TORSION THEORY

General Problem.—The solution of the torsional properties of a section of any shape consists primarily in determining the distribution of lateral shearing stresses over the cross-section. The shear components will be of uneven distribution, except in the case of the circular section, and as a result plane sections will be warped during twisting as shown in Fig. 1. (Note that, in Fig. 1(*c*), the web and each flange are warped as individual rectangles, in addition to the warping of the section as a whole.)

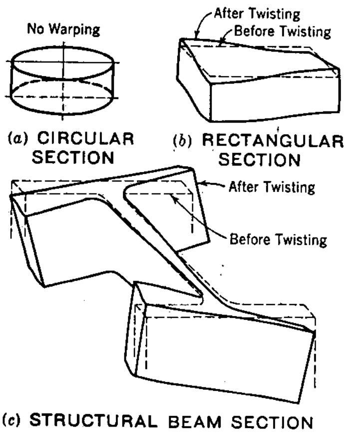

FIG. 1.—TWISTING OF BARS OF VARIOUS CROSS-SECTIONS

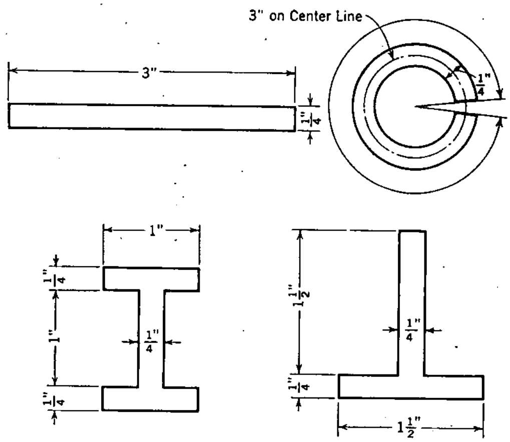

FIG. 2.—SECTIONS HAVING APPROXIMATELY EQUAL TORSIONAL RIGIDITY

It is assumed that the lateral displacements are proportional to the angular twist and to the distance from the twisting axis (as is the case in a circular section). The longitudinal displacements cause the warping, and the resulting distribution of shearing stress is taken care of by introducing a "stress-function," F, of x and y. This function must satisfy the differential equation:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -2\,G\,\theta \qquad (1)$$

in which, G = the shearing modulus of elasticity, and θ = angle of twist, in radians per inch. It may be shown that the function, F, must be a constant along the boundary of the section for solid bars, and, therefore, may be chosen arbitrarily as equal to zero.

If the boundary conditions are such that Equation (1) may be solved and the value of F determined, it is possible to evaluate the torsion constant of the section and find the stress at any point in the cross-section. Formulas for the torsion constant and critical shearing stresses have been derived in this manner for such sections as the square, rectangle, ellipse, equilateral triangle, and sector of a circle (1).

In the case of the circular shaft the shearing stress components have a uniform distribution along each radius, and since the longitudinal shear is likewise evenly distributed, there is no longitudinal warping of first-order importance. The well-known simple theory using the polar moment of inertia is thus applicable to the case of the circular section.

If, in some way, warping which takes place in non-circular sections is restrained or prevented, longitudinal fiber stresses will be introduced and the beam stiffened and strengthened.

The Membrane Analogy.[4]—Equation (1) may be solved mechanically for any cross-section by means of Prandtl's membrane analogy, thereby overcoming the mathematical limitations of the theoretical derivation.

In the application of this analogy, a soap film is stretched across an opening having the same shape as the structural section under consideration. The bubble is distended slightly by a variation in pressure. Prandtl showed that the following relations obtain for this bubble: (1) The torsion constant, K, is proportional to the total volume of the displaced bubble; (2) the shearing stress at any point is proportional to the maximum slope of the film at that point; and (3) the contour lines on the bubble give the direction of maximum shearing stress.

The analogy is also useful as an aid in visualizing the rigidity and stress distribution in various sections, and makes evident why the four sections shown in Fig. 2 have approximately equal rigidities in pure torsion, since the volumes of the various soap bubbles are approximately the same in each case. This would not be the case if the ends of the beams were restrained.

Evaluation of the Torsion Constant

Definition.—The torsion constant, K, is the measure of the torsional rigidity and twisting deflections. It is also a part of any formula for torsional shearing stresses, and may be determined from test results by observing the ratio of torsional moment to unit twist, in radians per inch, at any place below the yield point of the beam, and dividing this ratio by the shearing modulus of elasticity.

The Relation Between K and J.—When a torsional couple, T, is applied to a circular shaft of radius, r, the maximum shearing stress, τ, at the surface, is given by:

$$\tau = \frac{T\,r}{J} \qquad (2)$$

in which, J = polar moment of inertia. In terms of T Equation (2) may be re-arranged to read:

$$T = \frac{\tau\,J}{r} \qquad (3)$$

The torque, T, may also be expressed in terms of θ and G, thus:

$$T = J\,G\,\theta \qquad (4)$$

[4] A detailed description of the soap film studies is given in a thesis by Bruce G. Johnston, Jun. Am. Soc. C. E., presented to Lehigh University in partial fulfillment of the requirements for the degree of Master of Science.

For non-circular sections the torsional resisting moment may again be expressed in terms of θ and G, with the substitution of K, the torsion constant, in place of J, thus:

$$T = K\,G\,\theta \qquad (5)$$

The torsion constant, K, is equal to the polar moment of inertia for circular sections. Although for non-circular sections it is always less than the polar moment of inertia, there is no direct relation between the two factors.

The Rectangle.—In dealing with structural shapes, two principal types of section require consideration, the rectangle, and the rectangle modified by sloping sides, as in the flange of a standard **I**-beam. In the case of the rectangle an accurate formula was derived originally by Saint Venant (1):

$$K = \frac{n^3\,b}{3} - 2\,V\,n^4 \qquad (6)$$

in which n = the breadth of a rectangular section; b = the length of a rectangular section; and V = a factor depending upon the ratio, $\frac{b}{n}$, but practically constant for $\frac{b}{n} > 3$. Fig. 3 shows the values of V for $\frac{b}{n}$ from 1 to 3. For $\frac{b}{n}$ -ratios greater than 3, $V = 0.105$, and for $\frac{b}{n}$ greater than 4, $V \doteq 0.10504$. Equation (6) finds a direct, qualitative interpretation in the

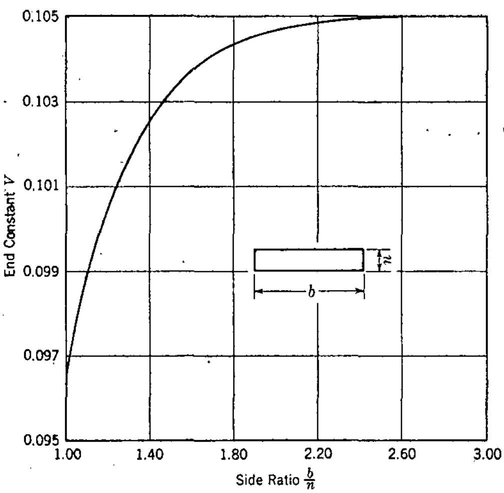

FIG. 3.—END CONSTANTS FOR RECTANGULAR SECTIONS, WITH $\frac{b}{n} < 3$. (SEE EQUATION (6)).

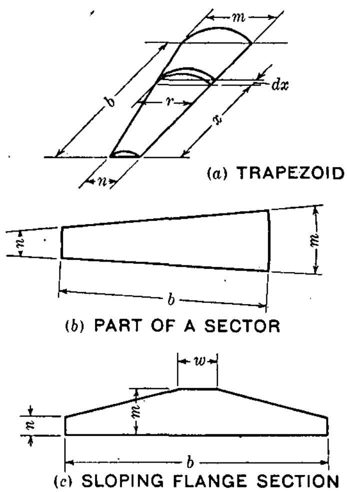

FIG. 4.

soap film analogy. It is evident that for long rectangular sections the bubble will be of constant cross-section along the central part, but at the two ends it will be contracted and brought down to meet the small side. The quantity, $-2\ V\ n^4$, then represents the "end loss," which for long sections is evidently a function of n only.

It also follows that if the ends were made discontinuous, as if they were parts of infinitely long rectangles, one might state without error:

$$K = \frac{1}{3}\ n^3\ b \qquad (7)$$

and for any differential length, dx, along the section:

$$K = \frac{1}{3}\ n^3\ dx \qquad (8)$$

The Section with Sloping Sides.—Equation (8) provides a basis for evaluating K for the sloping flange section. Considering the section shown in Fig. 4(a), let the thickness at any point be taken as r. Then, if the ends are assumed discontinuous:

$$K = \frac{1}{3} \int_0^b r^3\ dx \qquad (9)$$

Evaluating r in terms of m and n, and integrating:

$$K = \frac{b}{12}\ (m + n)\ (m^2 + n^2) \qquad (10)$$

in which m = major flange thickness and n = minor flange thickness. A deduction must be made for end effects, as in the case of the simple rectangle, thus:

$$K = \frac{b}{12}\ (m + n)\ (m^2 + n^2) - V_L\ m^4 - V_S\ n^4 \qquad (11)$$

in which V_L and V_S are the end constants, V, for the large end and the small end, respectively, of the flange (see Fig. 5). The evaluation of these two constants was the work of Professor J. B. Reynolds, through an analysis of a section having the shape shown in Fig. 4(b):[5]

$$V_L = 0.10504 - 0.10000\ S + 0.08480\ S^2 - 0.06746\ S^3 + 0.05153\ S^4 \qquad (12)$$

and,

$$V_S = 0.10504 + 0.10000\ S + 0.08480\ S^2 + 0.06746\ S^3 + 0.05153\ S^4 \qquad (13)$$

in which S = the total slope of the section; that is, $\dfrac{m - n}{b}$.

Torsion Constant for **H**-*Beams and* **I**-*Beams.*—The foregoing supplies a basis for evaluating the K-values of the component parts. Taking the section shown in Fig. 4(c) as a basis for the sloping flange section, the sum of two trapezoids and a small rectangular part is expressed by:

$$K_f = \frac{b - w}{12}\ (m + n)\ (m^2 + n^2) + \frac{1}{3}\ w\ m^3 - 2\ V_S\ n^4 \qquad (14)$$

[5] "Theory of Elasticity," by A. E. H. Love, Fourth Edition, p. 319.

in which K_f = the K-value for the flange. The web is considered as a discontinuous section between the flanges, giving:

$$K_w = \frac{1}{3} (d - 2\,m)\, w^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

in which K_w = the K-value for the web; d = total depth of beam; and, w = thickness of web (see Fig. 6). There still remains the evaluation of the added rigidity due to the connection of the flange and web and also due to the fillet at this point. It is evident that these will cause a considerable "hump" in the soap bubble.

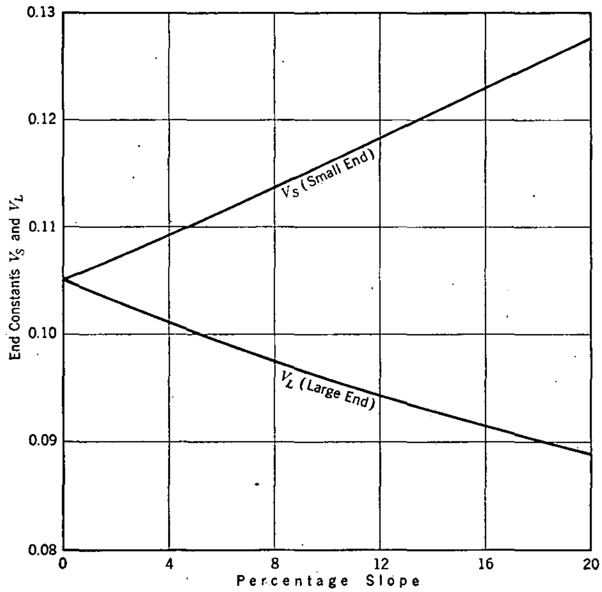

FIG. 5.—END CONSTANTS FOR K-VALUES OF FLANGES WITH SLOPING SIDES.

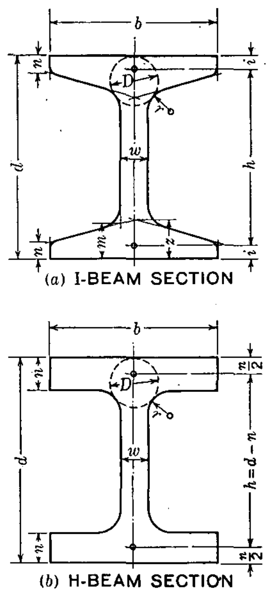

FIG. 6.

Trayer and March (4) in an investigation of aircraft strut sections assumed this addition to the torsion constant to be proportional to the fourth power of the diameter of the largest circle that can be inscribed at the juncture of the web and flange (see Fig. 6), giving, as an additional K-value:

$$K = \alpha\, D^4 \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

in which D = the diameter of an inscribed circle; and α = a factor that depends on two ratios, $\frac{w}{m}$ and $\frac{r}{m}$.

Values of α for sections with parallel-sided flanges, and for sections with flange slopes of 1 on 6, are given in Fig. 7. These curves were determined experimentally as the result of soap film tests which will be described subsequently. It will be noted in Fig. 7(a) that for parts of the curves to the right of *a-a*, the lines are parallel and uniformly spaced. All standard rolled beams are in this area, in which case, for parallel flange sides:

$$\alpha = 0.094 + 0.070 \frac{r}{m} \quad \text{(17)}$$

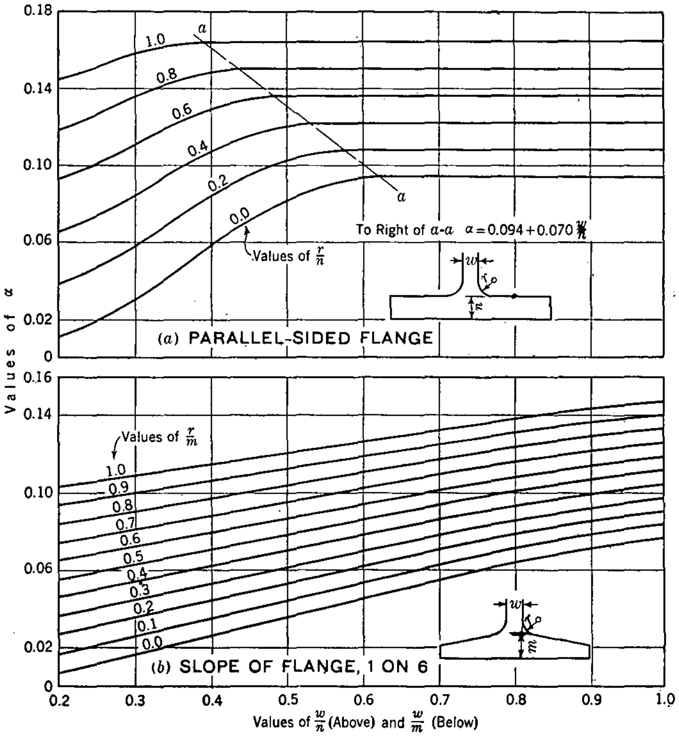

FIG. 7.—VALUES OF α FOR RIGHT-ANGLE JUNCTION OF FLANGE AND WEB.

For flange sections with side slopes of 1 on 20 and 1 on 50 the following formulas represent an interpolation between the curves in Fig. 7:

For a slope of 1 on 20:

$$\alpha = 0.066 + 0.021 \frac{w}{m} + 0.072 \frac{r}{m} \quad \text{(18)}$$

and, for a slope of 1 on 50:

$$\alpha = 0.084 + 0.007 \frac{w}{m} + 0.071 \frac{r}{m} \quad \text{(19)}$$

The various elements entering into the total K-values, can now be summarized as follows (refer to Fig. 6):

For sloping flange sections,

$$K = \frac{b - w}{6} (m + n) (m^2 + n^2) + \frac{1}{3} (d - 2 m) w^3 + 2 \alpha D^4 - 4 V_S n^4 \quad \text{(20)}$$

Values of V_L and V_S for standard slopes are, as follows:

S	V_L	V_S
$\frac{1}{6}$	0.09045	0.12441
$\frac{1}{20}$	0.10026	0.11026
$\frac{1}{50}$	0.10307	0.10707
$\frac{1}{\infty}$	0.10504	0.10504

and, for sections with parallel-sided flanges,

$$K = \frac{2}{3}\, b n^3 + \frac{1}{3}\,(d - 2\,n)\, w^3 + 2\,\alpha\, D^4 - 0.42016\, n^4 \quad \ldots\ldots (21)$$

SECTIONS	RIGIDITY	STRENGTH
(a)	100.0	100.0
(b)	637.0	332.0
(c)	5.5	18.0
(d)	70.0	62.0
(e)	88.0	74.0
(f)	341.0 (Approx.)	280.0 (Approx.)
(g)	9.9 (Nearly Exact)	22.2 (Approx.)
(h) $r = w = \frac{2}{3}\,n$	11.6 (Nearly Exact)	22.8 (Approx.)
(i) $6\frac{1}{2}''$ Fixed Ends 1'6"	78.1 (Approx.)	38.3 (Approx.)

FIG. 8.—TORSIONAL RIGIDITY AND STRENGTH OF DIFFERENT SECTIONS OF EQUAL CROSS-SECTION AREAS.

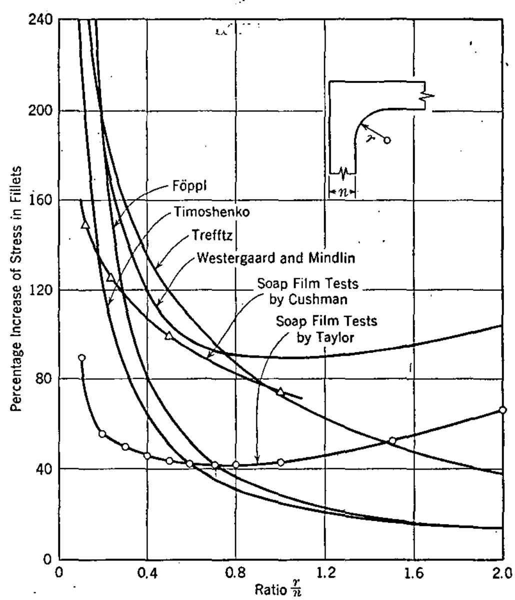

FIG. 9.—FORMULAS FOR STRESS CONCENTRATION IN FILLETS DUE TO TORSION.

Diameter D may be determined by a large scale layout, or by the following formulas:

For parallel-sided flange sections,

$$D = \frac{(n + r)^2 + w\left(r + \frac{w}{4}\right)}{2\,r + n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

for sloping-sided flange sections,

$$D = \frac{(B + z)^2 + w\left(r + \frac{w}{4}\right)}{B + r + z} \qquad \text{(23)}$$

in which z = the maximum flange depth shown in Fig. 6, and,

$$B = r\,S\left[\sqrt{\frac{1}{S^2} + 1} - 1 - \frac{w}{2\,r}\right] \qquad \text{(24)}$$

Comparative Efficiencies of Different Sections.—Fig. 8 shows the comparative torsional efficiencies of certain different shapes, illustrating the striking advantages of the hollow box, or tubular, construction. These advantages would not obtain entirely if the section were built up by use of bolts or rivets.

Examples of Design

General Statement

Structural members are often required to carry torsional loads, generally as a secondary factor combined with bending or direct stress. Problems involving the bending of unsymmetrical sections, and problems of elastic stability, such as the buckling of flanges during bending, also require a knowledge of torsional properties.

In the design of short beams to carry torsional loads considerable advantage may be obtained by fixing the ends, resulting in increased strength and rigidity, with corresponding decrease of angular deflection. External fixity is not needed—it is only necessary to box in the two flanges at each end and thereby prevent, as nearly as possible, their relative warping. The two flanges then act as two rectangular fixed-ended beams carrying a lateral displacement, mutually opposed in direction.

In designing long beams, the end effect tapers out rapidly toward the center, and the formulas for pure torsion, free-ended, will be adequate and simpler in their application. It would still be of practical advantage and a means of additional safety, however, to "box" the ends of the beam.

Free-Ended Torsion

If a beam is to be designed as free-ended, only shearing stresses need be considered. Regardless of the partial restraint that does exist as an incidental feature of the details, such a design will be on the safe side. The critical shearing stresses will occur along the outer surface of the beam where the material is thickest, generally along the outside center line of the flange and along the inside re-entry fillets.

The shearing stress is a function of the thickness of the material and the following formula is proposed for the maximum shearing stress in the flange of an **H**-beam or an **I**-beam in free-ended torsion:

For parallel-sided flange sections,

$$\tau_f = \frac{T\,(D + n)}{2\,K} \qquad \text{(25a)}$$

for sloping flange sections,

$$\tau_f = \frac{T\,(D + m)}{2\,K} \qquad (25b)$$

and for shear stress in the web,

$$\tau_w = \frac{T_w}{K} \qquad (26)$$

in which T = torsional moment. Although Equation (26) is in accord with torsional theory it gave results which proved low by comparison with actual tests. The following tentative formula was found to give better agreement:

$$\tau_w = \frac{T\,(w + 0.3\,r)}{2\,K} \qquad (27)$$

Equation (27) was adopted for use in the reported investigation. More accurate stress formulas might be developed by further use of soap film tests, taking slope measurements at critical points, and testing sections with various ratios of web, fillet, and flange as was done in determining the torsion constant. An equation similar to Equation (27) will give practically the same values for flange stress as Equation (25), thus:

$$\tau_f = \frac{T\,(n + 0.3\,r)}{K} \qquad (28)$$

Equation (25) was used in the reported investigation.

Concentration of Shearing Stress at Re-Entry Fillet

At the re-entry fillet of an **I**-beam torsional shearing stress concentrations occur due to the sharp curvature of the fillet. These stresses are mostly of a local nature and do not greatly influence the yielding of the beam as a whole. Although they do not necessarily govern the design of the beam they are important in the study of adequate fillet sizes and in the determination of loads producing strain lines in the fillets. In a beam subjected to flexural as well as torsional stresses the total stress concentration at the fillet is the sum of the stresses due to these two causes.

The concentration of torsional stress at the fillet between flange and web is illustrated in Fig. 9 for the formulas developed by Föppl(9), Trefftz(8), Timoshenko(5), and Westergaard and Mindlin, and for soap film experiments conducted by Griffith and Taylor(3), and by Cushman(13). The analysis resulting in the Westergaard-Mindlin curve was communicated to the writers by its originators. It is noted that all curves indicate a rapid increase in stress concentration with the decrease of the ratio, $\frac{r}{n}$, between the radius of the fillet and the thickness of the section next to it, particularly for ratios of less than 1.0. Strain lines, therefore, will appear at relatively small torsional moment for sections having small fillets.

As the fillets become increasingly large another factor is introduced since the increased stress due to the greater thickness of material becomes of more importance. The rigidity of the beam is increased in proportion to the third power of the thickness, whereas the stress varies directly with the thickness. Hence, for any given moment the stress would actually decrease. The curve of soap film tests by Griffith and Taylor indicate this reverse effect, but further experiments along this line should be conducted to establish these relations more definitely.

Fixed-Ended Torsion

Assumptions.—If the ends of a non-circular section are fixed in some manner so as to prevent free warping of the end section, pure torsion no longer exists.

Various investigators have studied this problem—Föppl(9), Timoshenko(5), Sonntag(7), and others. In 1930 an investigation on channel sections was reported by Seely, Putnam, and Schwalbe (10). These investigators have generally been concerned with the problem of elastic equilibrium involved in the side buckling and twisting of a beam in bending without lateral support. Hence the torsion problem has been a secondary issue.

In the present tests both ends of structural beams have been fixed by welding on heavy end plates and additional side stiffening plates between the flanges at each end of the beam. The major stiffening effect produced is that of fixing the flanges relative to each other. (Refer to Fig. 10.) The

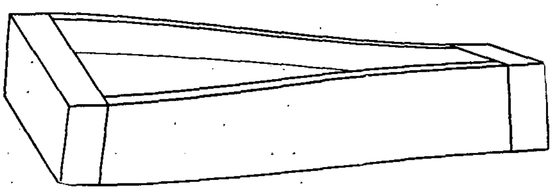

FIG. 10.—BEAM WITH FIXED ENDS.

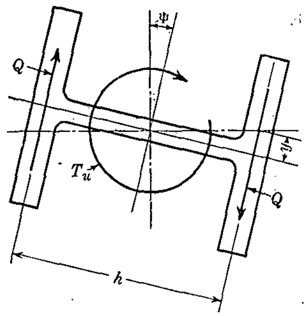

FIG. 11.—TWISTING OF A STRUCTURAL SHAPE.

prevention of the individual warping of the component rectangular parts would taper out so rapidly that it would be of negligible importance; but, by fixing the flanges with respect to each other, the effect of two opposed fixed-ended beams is produced, as illustrated in Fig. 10.

The following assumptions have been made: (1) The flanges remain at right angles to the web; (2) the angular deflection is small compared with the length of the beam; (3) the bending of each flange about its weaker axis is a negligible factor; (4) a point on the neutral axis of one flange can be located with sufficient accuracy by co-ordinates (x, y) measured along the perpendicular to the original position of the axes; (5) the two ends of the beam are held between mutually parallel planes as twisting takes place; (6) the displacement of the flanges due to beam action is due to bending only (that is, lateral shearing deflection is neglected; a correction is made afterward for shearing deflection in very short beams); and (7) $\frac{I_y}{2}$ = moment of inertia of one flange about the web axis.

Assumption (7) is quite accurate and of great convenience in the case of standard **I**-beams and **H**-beams for which I_y is given in all handbooks. The following symbols in addition to those previously given will be used:

$$a = \frac{h}{2}\sqrt{\frac{EI_y}{KG}}$$

or, for steel,

$$a = 0.806\, h\sqrt{\frac{I_y}{K}}$$

and for sloping flange sections,

$$h = d - \frac{2}{3}\left(n + \frac{z^2}{z + n}\right)$$

in which h = distance between flange centroids; or, approximately,

$$h = d - \frac{m + n}{2}$$

which is sufficiently accurate for practical purposes.

Derivation of General Equations.—Consider the equilibrium of any cut section as shown in Fig. 11. The outer torsional moment must be resisted by the internal moment of the resisting forces, thus:

$$T = T_u + Qh \qquad (29)$$

in which T_u = torque required to twist a beam in a free-ended condition; and Q = total lateral shearing force developed by one flange. In terms of the chosen co-ordinates, $\frac{h}{2}\, d\psi = dy$, approximately, and the twist, ψ, per unit length $= \frac{d\psi}{dx}$. Hence,

$$T_u = K\,G\,\frac{d\psi}{dx} = \frac{2\,K\,G}{h}\,\frac{dy}{dx} \qquad (30)$$

Furthermore, assuming that the larger moment of inertia of one flange is equal to $\frac{I_y}{2}$,

$$\frac{E\,I_y}{2}\,\frac{d^3y}{dx^3} = -\,Q \qquad (31)$$

Considering Equations (29), (30), and (31):

$$\frac{E\,I_y}{2}\,\frac{d^3y}{dx^3} - \frac{2\,K\,G}{h^2}\,\frac{dy}{dx} = \frac{-\,T}{h} \qquad (32)$$

Differentiating with respect to x, and substituting $a = \frac{h}{2}\sqrt{\frac{EI_y}{KG}}$:

$$a^2 \frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 0 \qquad (33)$$

As a general solution of this differential equation:

$$y = A \sinh \frac{x}{a} + B \cosh \frac{x}{a} + C + Dx \qquad (34)$$

Torsion with Both Ends Restrained.—By proper evalution of the constants for the conditions obtaining in a beam fixed at both ends:

$$y = \frac{T\,h\,a}{2\,K\,G}\left(\cosh \frac{x}{a} \tanh \frac{l}{2\,a} - \sinh \frac{x}{a} + \frac{x}{a} - \tanh \frac{l}{2\,a}\right) \qquad (35)$$

in which l = length of the beam. For $x = l$, the total deflection equals,

$$y = \frac{T\,h\,a}{2\,K\,G}\left(\frac{l}{a} - 2 \tanh \frac{l}{2\,a}\right) \qquad (36)$$

The moment in each flange equals,

$$M = \frac{E\,I_y}{2}\,\frac{d^2\,y}{dx^2} = \frac{T\,a}{h}\,\frac{\sinh u}{\cosh \dfrac{l}{2\,a}} \qquad (37)$$

in which $u = \frac{l}{2\,a} - \frac{x}{a}$; and, furthermore, the shear in each flange equals,

$$Q = \frac{E\,I_y}{2}\,\frac{d^3\,y}{dx^3} = \frac{-T}{h}\,\frac{\cosh u}{\cosh \dfrac{l}{2\,a}} \qquad (38)$$

The longitudinal stresses along the outer fibers of the flanges will be given by:

$$\sigma = \frac{M\,c}{I} = \frac{M\,b}{I_y} = \frac{T\,a\,b}{h\,I_y}\,\frac{\sinh u}{\cosh \dfrac{l}{2\,a}} \qquad (39)$$

Critical stress will be either longitudinal stresses at the end of the beam, or the sum of the lateral and torsional shearing stresses at the center of the beam.

At the end of the beam, $u = \frac{l}{2\,a}$, and, consequently,

$$\sigma = \frac{T\,a\,b}{h\,I_y} \tanh \frac{l}{2\,a} \qquad (40)$$

and at the center of the beam, $u = 0$, giving,

$$Q_c = \frac{T}{h} \operatorname{sech} \frac{l}{2\,a} = \frac{T}{h \cosh \dfrac{l}{2\,a}} \qquad (41)$$

At the center of the beam both the lateral and torsional shearing stresses will be maximum in the middle of the flange.

By the simple beam theory, the lateral shearing stress will be: For parallel flange sections,

$$\tau = \left(\frac{3}{2}\frac{Q}{A_f}\right) = \frac{Q\,b^2}{4\,I_y} \quad \text{.................... (42)}$$

in which $A_f = A$ for one flange. For sloping flange sections,

$$\tau = \frac{Q\,b^2\,(2\,n + m)}{12\,m\,I_y} \quad \text{.......................... (43)}$$

For torsional shearing stress at the center of the beam, the approximate formulas proposed in Equations (25) and (27) are applied. Instead of K, the torsion constant, an equivalent constant must be introduced, which is measured by the rate of angular twist at the center of the beam.

To obtain this, differentiate Equation (35) with respect to x, giving the slope of the flange, $\frac{dy}{dx}$, at any point. The unit angular twist, $\frac{d\psi}{dx} = \frac{2}{h}\frac{dy}{dx}$, and from this relation and the substitution of $x = \frac{l}{2}$, an expression is derived for $\theta_c = \frac{d\psi}{dx}$ at the center; thus,

$$\theta_c = \frac{T}{K\,G}\left[\frac{\cosh\left(\frac{l}{2\,a}\right) - 1}{\cosh\left(\frac{l}{2\,a}\right)}\right] \quad \text{................. (44)}$$

This derivation (Equation (44)) has been based on the assumption (6) that deflection is due to bending only. As the beam is shortened, however, shearing deflection becomes of increasing importance and should be considered.

Timoshenko(6) has indicated a strain energy method for calculating the deflection due to shear when cross-sections are constrained from warping. By combining his result for the simple cantilever beam the correction, $1 + 2.95\,\frac{b^2}{l^2}$, is obtained for a fixed-ended beam with point of inflection at the center; that is,

$$\theta_c = \frac{T}{K\,G}\left[\frac{\cosh\left(\frac{l}{2\,a}\right) - 1}{\cosh\left(\frac{l}{2\,a}\right)}\right]\left(1 + 2.95\,\frac{b^2}{l^2}\right) \quad \text{.......... (45)}$$

and, denoting the "equivalent" torsion constant at the center by C_c:

$$C_c = \frac{T}{\theta_c\,G} = K\left[\frac{\cosh\left(\frac{l}{2\,a}\right)}{\cosh\left(\frac{l}{2\,a}\right) - 1}\right]\left[\frac{1}{1 + 2.95\,\frac{b^2}{l^2}}\right] \quad \text{....... (46)}$$

Using C_c as the measure of torsional shear developed at the center of the beam instead of K and combining Equations (25) with Equations (42) and (41), Equations (47) to (53) are derived for combined torsional and lateral shearing stresses, as follows:

Total Maximum Shearing Stress at Center of Beam (Along Center Line of Flange).—For parallel flange sections:

$$\tau = T\left[\frac{b^2}{4\,h\,I_y \cosh\left(\frac{l}{2\,a}\right)} + \frac{(D+n)}{2\,C_c}\right] \quad\text{.......... (47a)}$$

for sloping flange sections:

$$\tau = T\left[\frac{b^2\,(2\,n+m)}{12\,h\,m\,I_y \cosh\left(\frac{l}{2\,a}\right)} + \frac{D+m}{2\,C_c}\right] \quad\text{.......... (47b)}$$

and the stress in the web may be computed by:

$$\tau_w = \frac{T\,(w + 0.3\,r)}{C_c} \quad\text{.......... (48)}$$

Total Twisting Deflections of Fixed-Ended Beams.—Equation (46) provides an equivalent torsion constant based on the unit angular twist at the center of the beam. A measure of the total twisting deflection of the beam over the entire length is desired and can be obtained effectively by evaluating an average equivalent torsion constant, which will be denoted as C_A.

The expression for total angular twist is, then:

$$\psi = \frac{T\,l}{C_A\,G} \quad\text{.......... (49)}$$

For very short beams most of the deflection is that producing shear and C_c approaches C_A in value. Equation (36) is an expression for the total deflection of the flanges due to bending only. Constant C_A may be evaluated from Equation (36) in so far as bending deflections are concerned.

The ratio of $\frac{C_A}{C_c}$ reduces to the following expression:

$$\frac{C_A}{C_c} = \frac{\left(\cosh\frac{l}{2\,a} - 1\right)}{\left(\cosh\frac{l}{2\,a}\right) - \left(\frac{2\,a}{l}\sinh\frac{l}{2\,a}\right)} \quad\text{.......... (50)}$$

The graph of $\frac{C_A}{C_c}$ is given on Fig. 12, permitting the quick calculation of C_A after C_c is known. As the length approaches zero, C_A should approach C_c in value and the curve on Fig. 13 gives the reduction to be made in C_A for various $\frac{l}{b}$ -ratios in terms of a factor to be multiplied by $(C_A - C_c)$. As an example, let: $C_c = 6.00$ by Equation (46); $\frac{l}{2\,a} = 2.00$; and, $\frac{l}{b} = 5.00$.

From Fig. 12, $\frac{C_A}{C_c} = 1.42$; and, from Fig. 13, the reduction $= (0.42)\ (0.105) = 0.44$; or, $C_A = 6.00\ (1.42 - 0.044) = 8.28$.

Torsion with One End Fixed and One End Free.—Often, in cases of combined bending and torsion, one end of the beam will be relatively unrestrained while the other end is fixed. At the free end there will be no lateral shearing stresses in the flanges and the shearing stress formulas (Equations (25)) for free end torsion will apply. The evaluation of Equation (34) for these end conditions gives the following: For maximum bending moment in the flange at the unrestrained end:

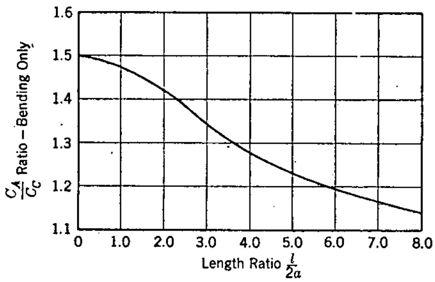

Fig. 12.—Positive Factor to Obtain C_c. To Be Used in Conjunction with Negative Factor from Fig. 13.

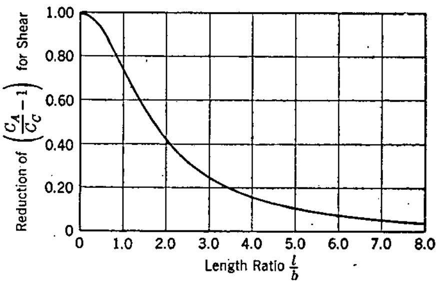

Fig. 13.—Reduction Factor for Shear (for Use with Fig. 12 to Obtain C from C_c).

$$M_{\max} = \frac{T\,a}{h} \tanh \frac{l}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (51)$$

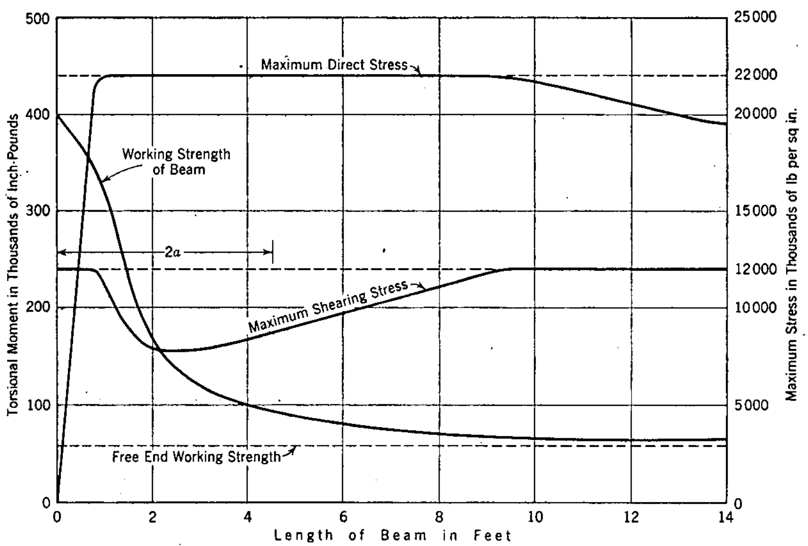

Fig. 14.—Working Strength of an 8 by 8-Inch Beam with Fixed Ends (Showing Limiting Stresses for Different Lengths).

and for maximum direct fiber stress in the outer edges of the flanges at the restrained end:

$$\sigma = \frac{T\,a\,b}{h\,I_y} \tanh \frac{l}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52)$$

The total angle of twist is given by:

$$\psi = \frac{T\,a}{K\,G}\left[\frac{l}{a} - \tanh\frac{l}{a}\right]\left[1 + 0.74\,\frac{b^2}{l^2}\right] \ldots\ldots\ldots\ldots (53)$$

Equation (53) will be accurate for all except very short beams.

Fig. 14 shows the application of the proposed formulas for maximum longitudinal and shearing stresses to varying lengths of an 8 by 8-in. **H**-beam fixed at both ends. It is noted that for lengths ranging between about 1 ft and 9 ft the longitudinal stresses based on an allowable stress of 22 000 lb per sq in. determine the design. For very short lengths and for long lengths the shearing stresses are critical.

Design of End Connection.—It is suggested that the end connections be built as illustrated in Fig. 15. Connections of this type proved very satis-

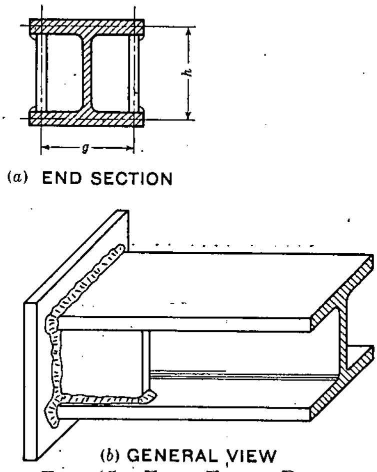

FIG. 15.—FIXED-ENDED BEAM.

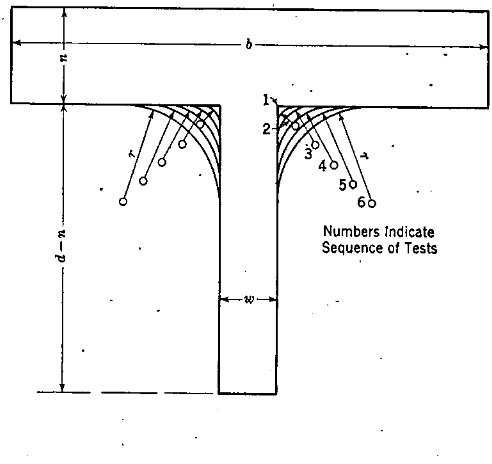

FIG. 16.—TYPES OF SECTIONS STUDIED BY SOAP FILM.

factory in the actual torsion tests and provided comparatively complete end fixity. The purpose of the end plates is to prevent relative warping of the flanges, and the following approximate analysis should serve as a guide in the design. The moment in one flange at the end is obtained from Equation (37)

$$M = \frac{T\,a}{h} \tanh \frac{l}{2\,a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54)$$

The value of l should be measured as the over-all length. Let Q = the total shear of the beam in the stiffening plate; and s = the distance between the stiffening plates. Then, if the stiffening plates alone are assumed to fix the ends of the beam:

$$M = Q\,s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55)$$

Substituting in Equation (54):

$$Q = \frac{T a}{h s} \tanh \frac{l}{2 a} \qquad (56)$$

The stiffener plate is welded to both flanges and to the end plate as well. The stiffener and the adjoining part of the end plate act as a short fixed-ended beam holding the flanges in place. No attempt was made to analyze the load distribution, and the design of the test beams was largely a matter of judgment.

The following tentative suggestions are made as the result of the tests:

(1) The length of the stiffener along the beam should be equal to about three-fourths the width of the flange for **H**-sections and to the full flange width for **I**-beams;

(2) The thickness of the stiffener plate material should be greater than that of the web thickness or greater than one-tenth the length of the stiffener plate along the beam;

(3) The stiffeners should be machined to a tight fit between the flanges and should be welded to flange and end plate continuously on the outer part;

(4) The end plates should have a thickness equal to twice the maximum thickness of the beam material; and

(5) The beam should be cut square and welded to the end plate with a continuous fillet weld about the entire beam end.

The stiffener plate and the weld between it and the flange should be designed to resist the shear as computed by Equation (52).

Design Examples

General Remarks.—The most economic structural **H**-beam or **I**-beam for torsional strength is one in which the material is most nearly of constant thickness throughout and is as thick and compact as obtainable. Column sections with parallel sided flanges and of the heaviest rolling in each series most nearly satisfy these requirements.

The torsional design should be made with ends assumed to be free in the case of riveted or bolted end connections; any percentage of end fixity incidentally present will simply provide an additional factor of safety. Only shearing stresses as computed by Equations (25) and (27) need be considered in free-end design.

Beams with boxed in and continuously welded end connections will be somewhat stiffer and stronger depending on the length. Both longitudinal stresses and shearing stresses must be considered (see Fig. 14). Tests indicate that the shearing stresses generally determine the yield point of the beam as a whole. The local direct stresses at the ends affect initially only a small part of the beam and are in the nature of secondary stresses. Shearing yield on the other hand occurs along the entire beam length. The allowable direct fiber stress will be made 22 000 lb per sq in. in the present discussion. It is suggested that allowable fiber stresses usual in secondary stress design be applied in general to these stresses.

General Data.—The following data apply to all the examples: Allowable working normal stress, $\sigma = 22\,000$ lb per sq in., for secondary stresses due to fixed-end torsion; allowable working shear stress, $\tau = 12\,000$ lb per sq in.; $E = 29\,000\,000$ lb per sq in.; $G = 11\,150\,000$ lb per sq in.; Poisson's ratio, $\mu = 0.30$; l = over-all length of a beam, including stiffeners, along which a uniform torsional moment is assumed to act; $1° = 0.01745$ radian; and 1 radian = 57.30 degrees. Computations in these problems were made with a 10-in. slide-rule.

Design Example A.—A long beam with torsional deflection limited: Assume a beam 20 ft long designed to resist a total torsional moment of 20 000 in-lb with maximum total twist under the load limited to 1.2 degrees. The procedure for designing the beam as free ended involves three steps: (1) Determine the unit angle of twist, θ, in radians per inch, thus, $\theta = \frac{1.2 \times 0.01745}{20 \times 12} = 0.0000872$ radian per inch; (2) calculate the required K-value from Equation (5), thus, $K = \frac{T}{G\,\theta} = \frac{20\,000}{11\,150\,000 \times 0.0000872} = 20.6$ in.[4]; and (3) refer to standard tables of K-values[6] and pick out the most economical section. In this manner, a Bethlehem section, 10 by 10 in., at 124 lb per ft (with $K = 20.37$) will be satisfactory. The end connection will provide additional rigidity and will allow a small tolerance in picking sections.

Design Example B.—Analysis of the torsional strength of a short beam (B8b, 8 by 8, at 67 lb per ft), with different end connections: The general data applying to this case are: $l = 66$ in., over-all; $K = 5.145$ in.[4]; $I_y = 88.6$ in.[4]; $n = 0.933$ in.; $b = 8.287$ in.; $D = 1.206$ in.; $h = 9.000 - 0.933 = 8.067$ in.; $a = 0.806\,h\sqrt{\frac{I_y}{K}} = 27.0$ in.; $\frac{l}{2\,a} = \frac{66}{54} = 1.22$; $\cosh \frac{l}{2\,a} = 1.8412$; and $\tanh \frac{l}{2\,a} = 0.8397$.

The free-ended working strength is computed by Equations (25) thus:

$$T = \frac{2\,K\,\tau}{D + n}\;\frac{(2)\,(5.145)\,(12\,000)}{1.206 + 0.933} = 57\,700 \text{ in-lb.}$$

To determine the fixed-ended working strength, based on shear, compute the equivalent torsion constant, C_c, for the center of the beam by Equation (46), thus:

$$C_c = 5.145\left(\frac{1.8412}{1.8412 - 1}\right)\left(\frac{1}{1 + 2.95\left(\frac{8.29}{66}\right)^2}\right) = 10.77 \text{ in}^4.$$

Then, from Equation (47),

$$T = \frac{12\,000}{\dfrac{8.29^2}{4 \times 8.067 \times 88.6 \times 1.841} + \dfrac{1.206 + 0.933}{2 \times 10.86}} = 108\,000 \text{ in-lb}$$

[6] See, for example, Bethlehem Manual of Steel Construction, Catalog S-47, 1934, p. 285.

To determine the fixed-ended working strength, based on longitudinal fiber stresses at ends (tension or compression), apply Equation (40):

$$T = \frac{22\,000}{\dfrac{27.0 \times 8.287 \times 0.8397}{8.067 \times 88.6}} = 84\,000 \text{ in-lb}$$

The longitudinal stresses, therefore, determine the design of the beam, and the allowable torsional moment is 84 000 in-lb.

The shear to be resisted by the end plate is computed by Equation (56); thus, if s is assumed as 6.5 in.,

$$Q = \frac{84\,000 \times 27.0}{8.067 \times 6.5} \times 0.8397 = 36\,300 \text{ lb.}$$

A $\frac{5}{8}$-in. plate fitted into the 7.13-in. space between the flanges and 6 in. in length along the beam, will satisfy the requirements suggested. Assume a $\frac{5}{8}$-in. fillet weld between the stiffener and the flange. If $s = 8.827 - 3 \times \frac{5}{8} = 6.41$ in.,

$$Q = \frac{6.5}{6.41} \times 36\,300 = 36\,800 \text{ lb}$$

The stress in the plate is:

$$\tau = \frac{36\,800}{\dfrac{5}{8} \times 6} = 9\,800 \text{ lb per sq in.}$$

and in the weld:

$$\tau = \frac{36\,800}{\dfrac{5}{8} \times 0.707 \times 6} = 13\,900 \text{ lb per sq in.}$$

which is too high. Make the plate $7\frac{1}{2}$ in. long rather than 6 in. Then the stress in the weld is:

$$\tau = \frac{6}{7.5} \times 139\,000 = 11\,100 \text{ lb per sq in.}$$

which is less than the limit of 11 300 lb per sq in. recommended by the American Bureau of Welding.

Test Results

Soap Bubble Tests: Purpose and Program

The purpose of the series of tests reported herein was to evaluate, accurately, the torsion constant of structural **H**-beam and **I**-beam sections. Specifically, this problem narrowed down to determining the added torsional rigidity introduced by the juncture of two rectangles, with fillets at the re-entry corners, in excess of the torsional rigidity of these rectangles treated as separate members. The problem, therefore, was to determine α in Equation

(16). In order to establish the value of α for any shape of section it was necessary to consider two variables, $\frac{w}{n}$, the ratio of web to flange thickness, and $\frac{r}{n}$, the ratio of fillet radius to flange thickness. Furthermore, it was essential to study sections with sloping flanges as well as those with parallel sides.

A program of tests was outlined to cover a wide range of the two variables, $\frac{w}{n}$ and $\frac{r}{n}$. Although it would have been desirable to measure the slopes of the bubbles and thereby study the stresses, particularly in the fillets, such a study would have greatly reduced the total number of tests possible. It was thought better to establish the torsion constant definitely, in which case it was only necessary to measure the volume of the displaced bubble. In each series a basic web and flange thickness was adopted and after testing the section with zero fillet radius, the various fillets were cut away in sequence as shown in Fig. 16.

The curves obtained from the test results are shown in Fig. 7. It was found that scattered points occurred along the lines of 0.0 and 0.2 fillet radius. These variations may be due to the difficulty in machining the plates with the small fillets, a slight inaccuracy causing considerable variation in the diameter, D, of the inscribed circle. A further difficulty is encountered in the plates of zero fillet radius due to the tendency of the soap film to jump across these sharp corners. Most of the structural beams actually rolled have ratios of $\frac{w}{n}$ and $\frac{r}{n}$ both greater than 0.5 and in this area the data were quite consistent.

Tests of Steel Beams: Purpose and Apparatus

Torsion tests were made on steel beams, with several objects in view. The sections themselves were chosen so as to give a range of shapes and sizes as great as possible, and tests of certain unusual shapes, which are not at present standard, were nevertheless valuable in the investigation.

Tests of free-ended conditions were made on a number of beams in order to check the results of the soap film experiments and the corresponding method of calculating the torsion constant. In these tests the distribution of shearing stress was studied, and a check was obtained on the proposed approximate formulas for stress. Tests of fixed-end conditions were made on different shaped beams and the effect of variations in length was studied. A type of end-connection design was developed to give a considerable degree of fixity.

A standard torsion machine of 26 000 in-lb capacity was used to test 3-in., fixed-end **I**-beams, of various lengths. It was also used for torsion tests of round bar samples of all material to determine the shearing modulus of elasticity.

The cable torsion rig, shown in Fig. 17, with an ultimate capacity of 750 000 in-lb was used in the major tests.[7] Most of the large beams were tested in lengths of 6 ft, but two tests each were made on beams 1 ft 3 in. and 3 ft long by means of the same sheaves and cables adapted for use with

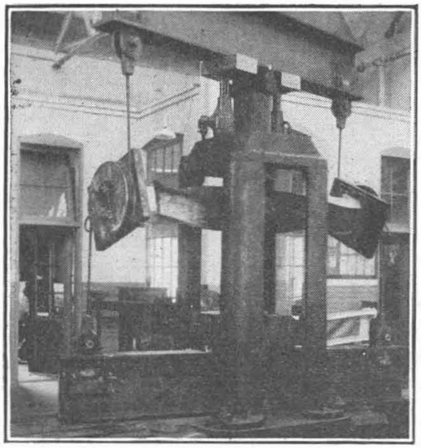

FIG. 17.—THE CABLE TORSION RING.

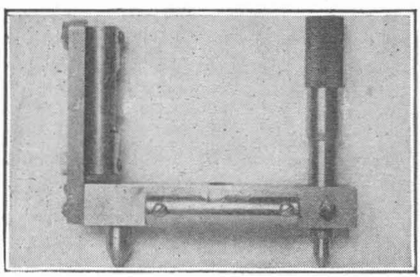

FIG. 18.—LEVEL BAR.

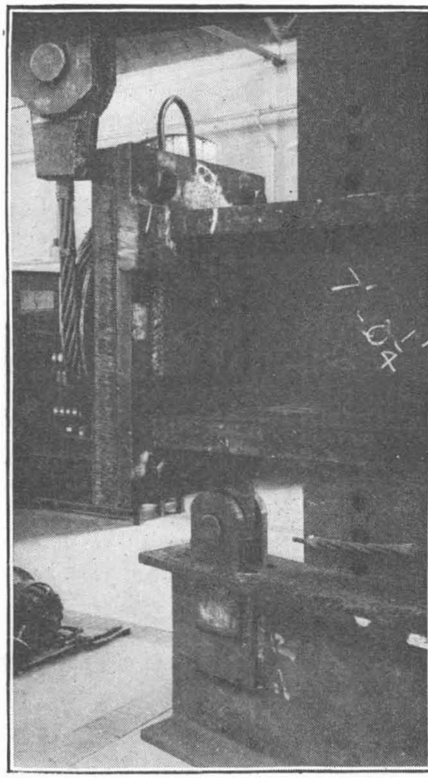

FIG. 19.—CONNECTION FOR FREE-ENDED BEAMS.

shorter top and bottom beams. During the tests of light beams, cables $\frac{5}{8}$ in. in diameter were used because of their flexibility and ease of handling, but in the tests of the heavier and shorter sections the cable was changed to 1 in. in diameter in order to develop the full capacity of the machine.

The sheaves were made of material 2 in. thick and were machined to a minimum diameter of 17 in. A hole bored through one of the diameters allowed continuous action and reversing of the cable without fouling or introducing bending moment. This machine gave perfect satisfaction in every respect and was easily set up and dismantled. During tests the apparatus was in such a state of balance that the heavy pulling beams could be easily tilted either way by hand while maintaining a heavy torsional load of the test specimen.

[7] "Torsion Testing Machine of 750 000-Inch-Pounds Capacity," by Bruce G. Johnston, Jun., Am. Soc. C. E., *Engineering News-Record*, February 28, 1935, p. 10.

Measuring Devices.—The level bar illustrated in Fig. 18 was built to measure the change in relative altitude of two points 3 in. apart. It was used to measure relative angle changes in all the beam tests. The micrometer vernier permitted readings to 0.0001 in. and the level bubble was sensitive to micrometer changes of about 0.0003 in. The total range of the instrument was about 9° 30′ ± from the orginal level position.

The torsion meter was used in the torsion tests of round bars in the 26 000-in-lb torsion machine to measure the angular twist over a 3-in. section of the bar. This device consisted of two steel collars attached by pointed thumb screws to the bar and provided with two 1:1 000 Ames dials for measuring the tangential displacements. The instrument was the same as that used and described in a previous investigation at the Fritz Laboratory.[8] Twenty tensometers were used to obtain the strains at all critical points during tests.

Test Procedure and Method.—In each torsion test, whether fixed or free-ended, there were three principal objectives: First, to learn as much as possible about the strain and stress distribution; second, to measure the torsional stiffness, or ratio of torsional load to angular twist; and third, to learn the useful torsional load-carrying limit of the beam as a whole.

The strain and stress distribution was studied in two ways: First, by tensometers which were sensitive to changes in strain corresponding to from 150 to 300 lb per sq in. in stress, depending on the model type; and second, the beams were whitewashed with a mixture of water and hydrated lime which showed the distribution and location of the first surface strain-slip lines.

In computing the stresses from the observed strains the same values of E, G, and μ, as those used in Design Example A, were adopted. The torsional stiffness was gauged by measuring the relative angle changes between two points along the beam by use of the level bar. In the free-ended tests the angle changes were measured over a 36-in. length along the center part of the beam. In the fixed-ended beam the unit angle change varied along the length and a measure of the average stiffness was obtained by measuring the relative rotation of the end plates and of points a short distance from each end where the reinforcement ended.

The yield point of the beam as a whole was obtained by a study of the torque-twist diagrams and a knowledge of the load when the first strain-slip lines appeared. In a few cases a definite drop of the beam was noted and, in such cases, this was taken as the yield point. In most of the tests the yield point was taken as the torsional moment corresponding to the point on the torque-twist diagram where the co-tangent of the slope was 1.5 times the value of the co-tangent of the slope of the straight part.

Test Results.—Table 1, Appendix III, gives a general summary of all the tests made, including dimensions of beams and computed K-values based on actual dimensions. The dimensions were obtained by means of micrometers

[8] "Shearing Properties and Poisson's Ratio of Structural and Alloy Steels," by Inge Lyse, M. Am. Soc. C. E., and H. J. Godfrey, *Proceedings*, A.S.T.M., 1933, Vol. 33, Pt. II.

and calipers. Readings were taken at a number of different places on the beam, and averages from them were used to calculate the weight of the beam per foot of length. Furthermore, the beams were actually weighed and any discrepancy between computed and actual weight was taken care of by adjusting the average measured dimension to give the actual weight.

Table 2, Appendix III, presents the physical properties based on tests of samples taken from the test beams. The tensile values are based on the average of two tests of American Society of Testing Materials standard tension test specimens (2-in. gauge length). The torsion tests and slotted-plate shear tests were made in the same manner as described in a previous investigation at the Fritz Engineering Laboratory.[8] In most cases, the tensile specimens and round-bar torsion specimens were cut from the material where the flange and the web join, as it is at this point that critical torsional stresses develop. The material for the slotted-plate test specimens was cut from the webs of the beams.

Free-Ended Tests.—Free-ended tests were made on seven beams. Each specimen was held in the torsion rig by two bolts at each end which passed with a loose fit through the web and through the two angles, the angles being welded to end plates which, in turn, were bolted to the sheave plates of the testing rig. Torsional moment was applied by means of lugs welded to the end plates which engaged the flanges of the test specimens. The flanges

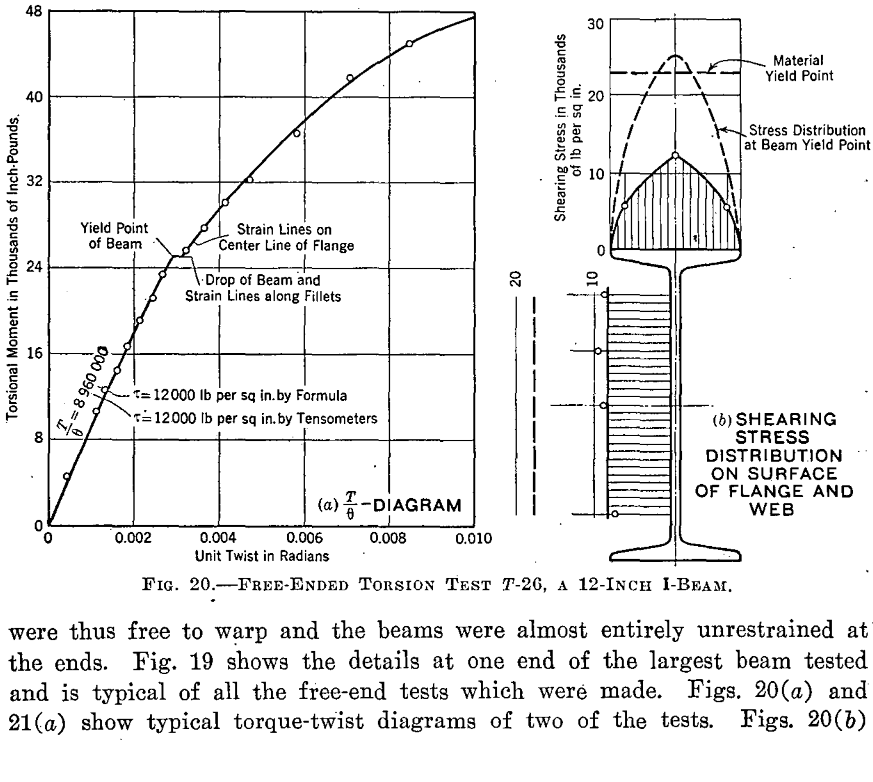

FIG. 20.—FREE-ENDED TORSION TEST *T*-26, A 12-INCH I-BEAM.

were thus free to warp and the beams were almost entirely unrestrained at the ends. Fig. 19 shows the details at one end of the largest beam tested and is typical of all the free-end tests which were made. Figs. 20(*a*) and 21(*a*) show typical torque-twist diagrams of two of the tests. Figs. 20(*b*) and 21(*b*) show the stress distribution based on tensometer readings taken in these same tests. The tensometers were placed on the flange and web surface at about the center cross-section of the beams and were set at an angle of 45° with the longitudinal axis of the beam, in order to measure

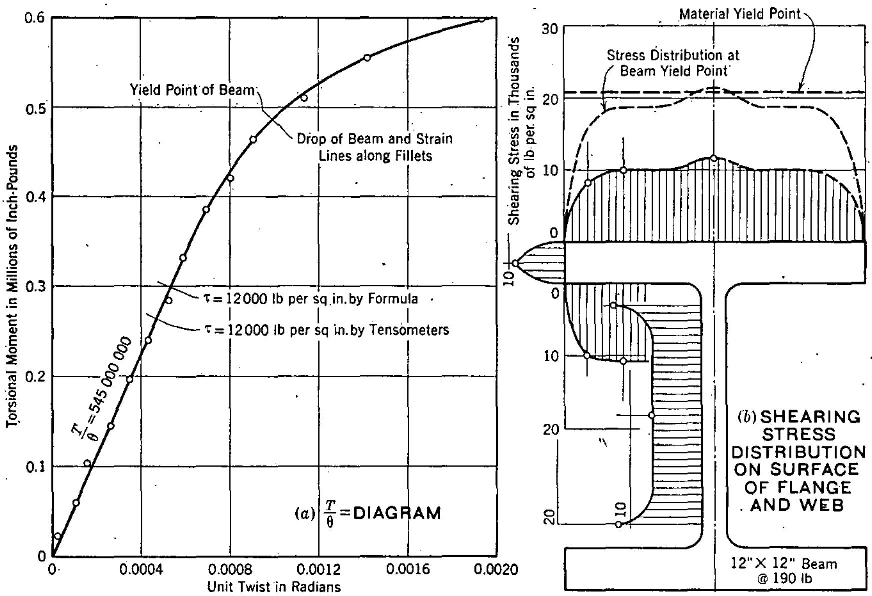

FIG. 21.—FREE-ENDED TORSION TEST *T*-31, A 12-INCH BETHLEHEM SECTION AT 190 POUNDS PER FOOT.

one of the principal strains which, in the case of pure shear, will be equal in both directions. In Figs. 20(*b*) and 21(*b*) the shaded areas show the stress distribution for $\tau_m = 12\,000$ lb per sq. in.

The data of actual stresses from tensometer readings were available at only a limited number of points. In drawing the curves of stress distribution these data have been supplemented by known facts, deducible from the general torsional theory, soap bubble tests, and from the actual beam tests; that is: (1) The shearing stress equals zero at outside corners; (2) there is a "hump" in the stress curve at the outside center of the flange, particularly if the fillets and the web thickness are relatively large as compared with the flange thickness; and (3) the shearing stress on the surface of the flange and web is approximately proportional to the thickness of the material.

Table 3, Appendix III, is a summary of the results obtained in the free-end tests. Good agreement is shown between the torsion constant computed from the measured dimensions and that obtained from the test results. The test value for K was obtained from the slope of the torque-twist diagram and from Equation (5); thus, $T = K\,G\,\theta$; or, $K = \dfrac{T}{G\,\theta}$.

The maximum variation for the K-value of test results was 6.7 and the average variation of seven tests was 2.26 per cent. It is noted that

K for the heaviest beam tested was about two hundred times greater than for the lightest beam and that the corresponding agreement for these tests was expressed by a variation of 0.9 and 0.0 per cent.

The shearing stress computed by Equation (25) gave average stresses 7% less than those based on tensometer readings. The stresses in the web by Equation (26) which, theoretically, should be correct, were much lower than as computed from tensometer readings. Equation (27) was suggested on the basis of these tests which, of course, are not complete enough to substantiate its adoption definitely. However, both Equations (25) and (27) are believed to be usable for practical design purposes with ordinary values of allowable shearing unit stress. The following special remarks apply to the individual free-end tests:

In Test T-12 strain lines appeared along the fillets at 13 500 in-lb; and along the outside center line of the flange at 15 200 in-lb. The yield point of the beam as determined by the slope of the torque-twist diagram was 15 900 in-lb.

In Test T-22 the freedom of the ends from restraint was checked by tensometers placed longitudinally near the ends. The strains were negligible. The first shear strain line along the center line of the flange appeared at 21 210 in-lb. At 25 610 in-lb, strain lines progressed rapidly along the flange and in the fillets, and a definite drop of the beam was noted.

In Test T-25 strain lines appeared along the fillets at 50 900 in-lb. Thereafter, the slope of the torque-twist diagram became nearly 50% greater than for lower loads, maintaining nearly the same slope up to 100 000 in-lb. The yield point was taken as 50 900 in-lb. Strain lines appeared along the outside of the flange at 77 340 in-lb.

In Test T-26 (see Fig. 20) a slight checking in the fillets was noted at 23 000 in-lb, with a drop-of-beam yield point at 25 000 in-lb; strain lines progressed along the outside of the flange at 26 500 in-lb.

In Test T-30 the yield of the beam was noted by the 1.5 on 1.0 slope of the torque-twist diagram at 64 000 in-lb. The first strain lines appeared over the web at an indeterminate load due to the presence of heavy scale on the section.

In Test T-31 (see Fig. 21) the beam and dropped strain lines occurred along the fillets at 48 800 in-lb. The yield point was noted by the 1.5 on 1.0 slope at 480 000 in-lb.

In Test T-33 strain lines occurred along the fillets at 65 600 in-lb. Yield point and strain lines appeared along the outside of the flange at 76 400 in-lb.

Fixed-End Tests.—Twenty-two fixed-end tests were made on beams of different sizes. Tests T-4 to T-13, inclusive, were made with different lengths of a 3-in., 7.5-lb I-beam. These specimens were welded to end plates, 1 in. thick, with a continuous $\frac{3}{8}$-in. fillet weld. The end plates were bolted to $1\frac{1}{4}$-in. plates which were attached to the jaws of the standard 26 000 in-lb torsion machine. The relative rotation of the end plates was measured by level-bar readings taken at each end. The remainder of the beams were tested in the cable-testing rig. All these beams were larger than the 3-in. I-beams

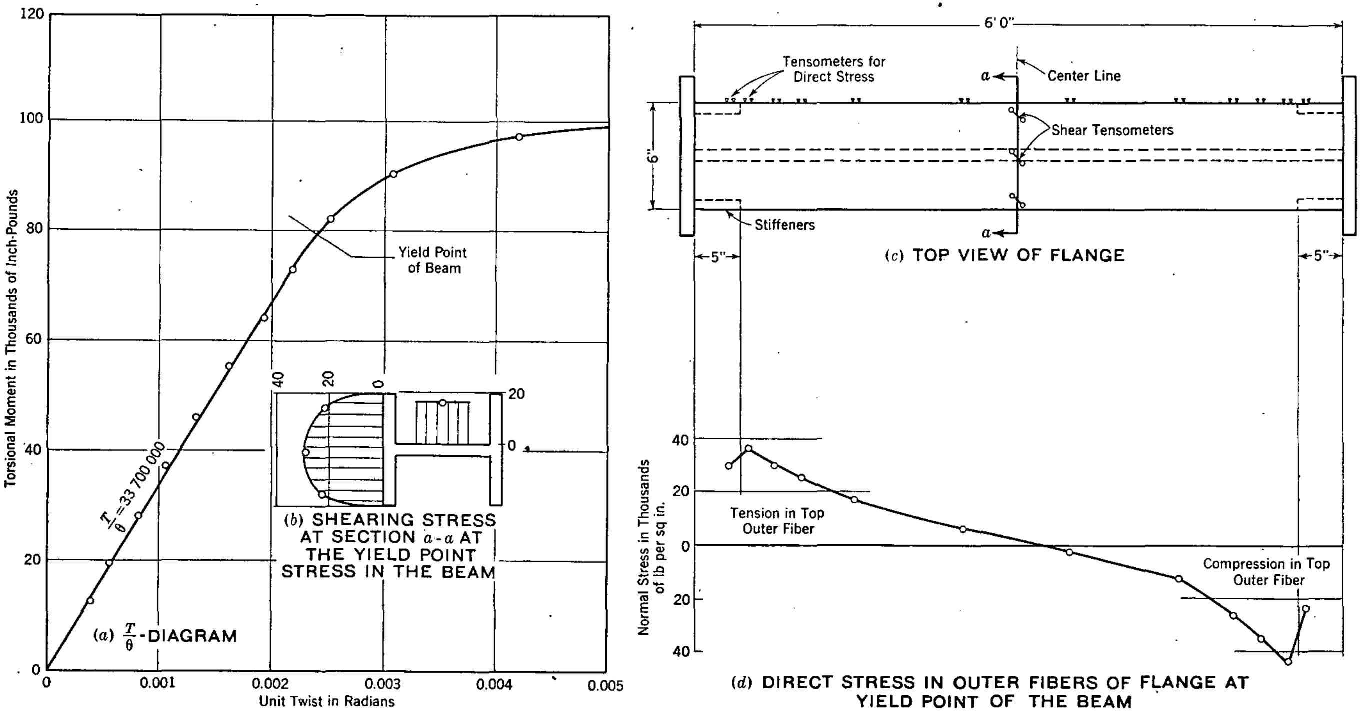

FIG. 22.—FIXED-ENDED TORSION TEST T-17, A 6-INCH H-BEAM AT 41 POUNDS PER FOOT.

and were welded to $1\frac{1}{4}$-in. end plates with additional end stiffeners, fitted and welded between the flanges, as was illustrated in Fig. 15.

Relative rotation of the end plates was observed on all beams by means of level-bar observations, and the twist of the large beams having stiffener plates was also measured at points just inside those plates. Strain readings for longitudinal and shearing strains were observed wherever feasible. Fig. 22(*a*) shows a typical fixed-end torque-twist diagram, and Fig. 22(*b*), 22(*c*), and 22(*d*) show the computed stresses from strain readings at the yield point of the beam. These observations are typical of all the fixed-end tests.

Table 4, Appendix III, gives the summary of the test results for fixed-end beams. In computing the values of C_c and C_A the question arose as to the correct length to be used. If 100% end fixity were possible the correct length would be slightly less than the over-all length and somewhat greater than the length between the end stiffener plates. However, the over-all length is the simplest approximation, and it gives the best results by comparison with the tests, except in the case of very short beams with end stiffeners. In these two tests (*T*-19 and *T*-27, Table 4) the apparent percentage of end-fixity seems inconsistently high. Two tests have unusually low percentages of end fixity (see Tests *T*-16 and *T*-24). The explanation for this is given under the special remarks. The average percentage of end fixity with Tests *T*-16 and *T*-24 omitted is 88.3 and all the 6-ft beams, except *T*-24, have an end efficiency greater than 85 per cent.

In most cases the yield points of the beams were determined from the slope of the torque-twist diagrams and the theoretical direct stresses computel on the basis of this yield-point torque are given in Column (14), Table 4 (Appendix III). It is noted that, in spite of incomplete end fixity, these stresses, in every case, are above the tensile yield-point strength of the material as given in Table 2 (Appendix III). Hence, all the beams

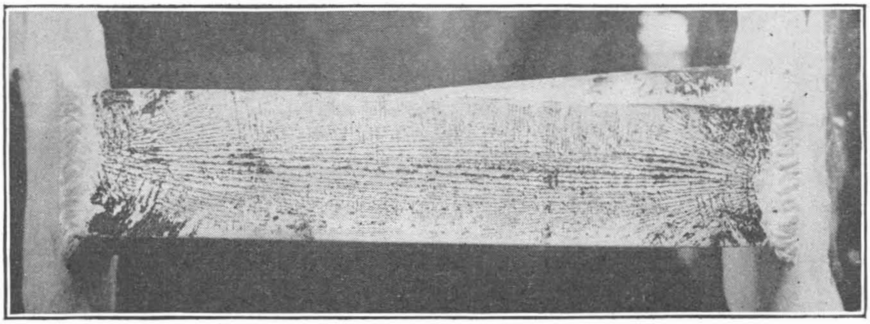

FIG. 23.—ILLUSTRATION OF STRAIN PATTERN.

would have been designed safely on the basis of working, direct, fiber stresses. The average of Column (14), Table 3 (Appendix III), is 55% more than the average yield-point strength of the material in the test beams.

The computed and measured shearing stresses in the flange agree well for all the 6-ft beams, with the exception of *T*-16 and *T*-24 in which the low

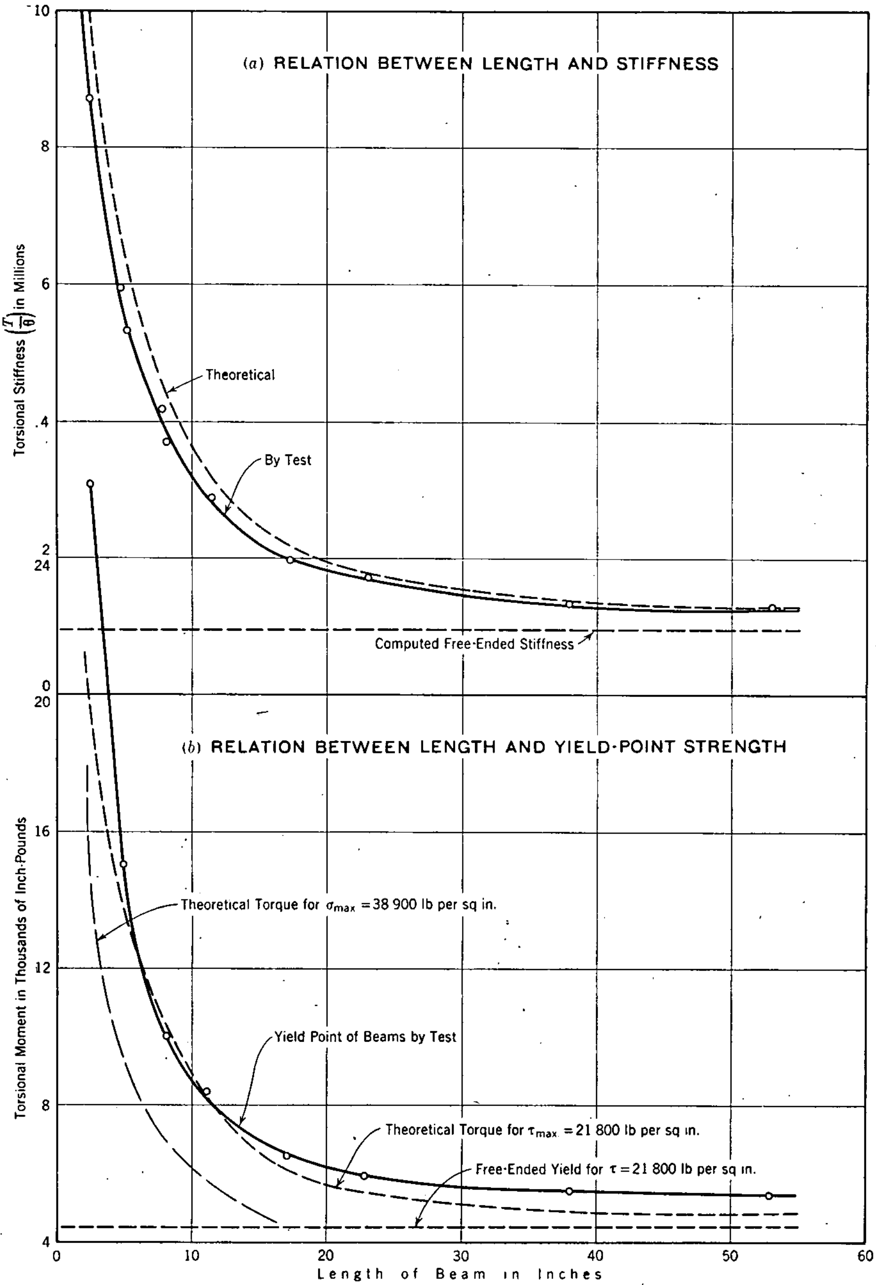

FIG. 24.—TEST OF 3-INCH I-BEAM AT 7.5 POUNDS PER FOOT, FIXED AT BOTH ENDS.

end fixity affects, directly, the shearing stress agreement. On the short 18-in. beams with stiffener plates (T-19 and T-27), the discrepancy between the computations and the test results is high, as might be expected. The computed stresses in the web give an approximate check on the stresses indicated by tensometers. Special remarks on fixed-end tests are, as follows:

Tests T-4 to T-12 were of different lengths of the same 3-in. **I**-beams and were well adapted to show the influence of end fixity on the strength and stiffness with the length of beam the only variable. The flange of each beam was whitewashed so that the appearance of first strain lines might be noted. Fig. 23 shows the strain line pattern on the flange of one of these beams after yield had taken place. Fig. 24(a) gives a graph of test results for this series showing the influence of length upon the rigidity, and Fig. 24(b) illustrates the influence of length upon the yield-point strength of the beams. Special attention in this series of tests was given to Tests T-8 and T-10. Stress measurements were taken along the extreme fiber of the flanges at short intervals of length and the lateral bending moment in the flanges for a definite torque load was computed from these readings. The bending moments along the beam were plotted and the curves were differentiated to give the lateral shear in the flanges. These results are compared in Fig. 25 with the theoretical variation in shear by Equation (38).

Tests T-15 and T-16 should be compared with the free-end test, T-14, of the same section. Test T-16 was a special run with additional stiffeners placed midway along the section. These stiffeners were of the same type as the end stiffeners and were parallel in a plane with the web. No outstanding stiffener was provided. Although this additional stiffener gave the beam an average stiffness 42% greater than Test T-15, it provided only 40.1% of the theoretical stiffness of a beam 3 ft in length, rigidly fixed at each end. The design of this beam would have been safe for strength, however, if based on the 3-ft length and designed for the proper longitudinal working stress.

Test T-17, in contrast to Test T-15, was of the heaviest 6-in. section rather than the lightest. It should be noted that an effective fixity of 96.8% was attained in this test. The end stiffeners were $\frac{5}{8}$ in. thick and 5 in. long. The torque-twist diagram and data on stress distribution are given in Fig. 22 as typical of the results for fixed-end tests.

Test T-18, of a subway column, provided an opportunity to observe a section of extreme proportions.

Tests T-19 to T-21, together with free-end Test T-22 on the same sized section, provided a series of different lengths of 8-in. **H**-section. It is noted that the shortest beam tested showed an end-fixity efficiency of 101%, whereas the general trend for shorter beams should be less end fixity because of the greater strains placed upon the end connection. This effect is explained by the fact that the over-all length was used in the computations. Although this is good approximation for the longer beams, the stiffness changes rapidly in the short length range and the correct length is some

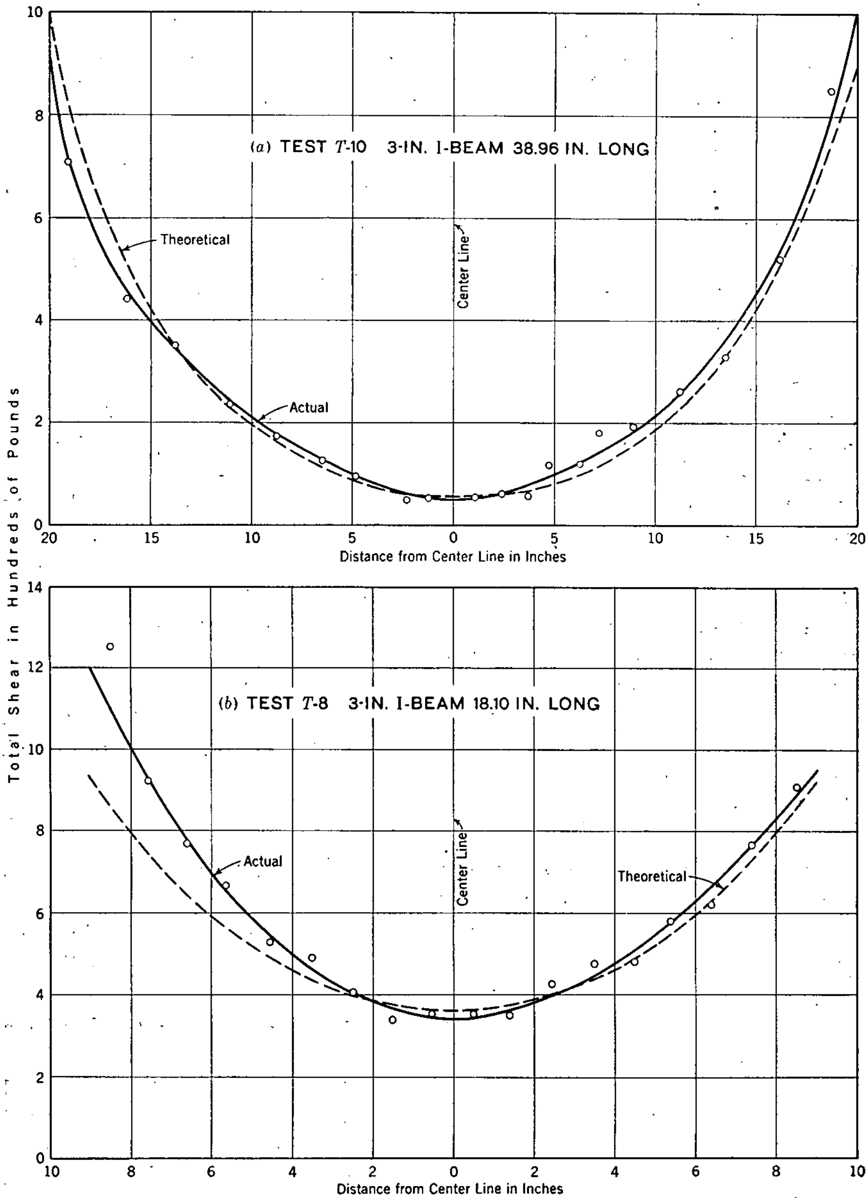

FIG. 25.—SHEAR DISTRIBUTIONS IN FLANGES OF FIXED-ENDED BEAMS FOR A TORSIONAL MOMENT OF 2 500 INCH-POUNDS.

value less than that used. Fig. 26 shows the strain-line distribution near the end of Beam T-21 after yielding had occurred.

Test T-23 is of the heaviest section of the 8-in. **H**-sections, whereas Tests T-19 to T-22 were of the lightest weight section.

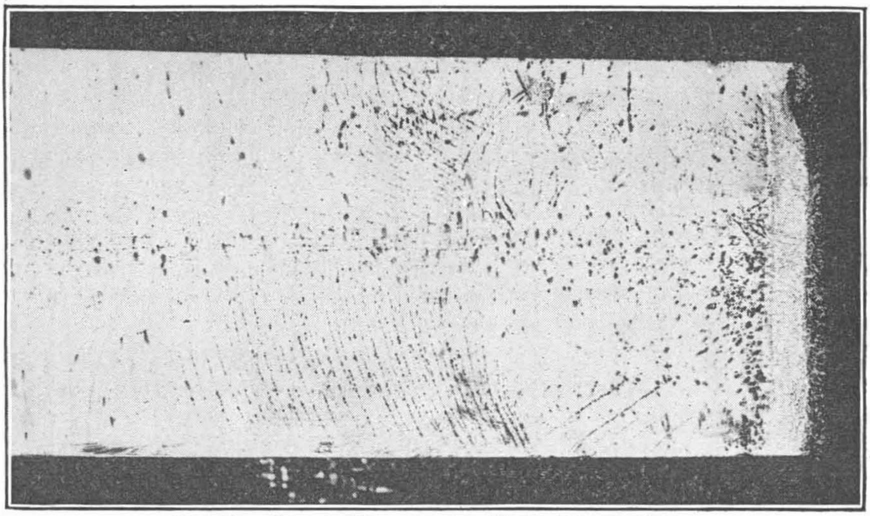

FIG. 26.—STRAIN LINES AT THE FIXED END OF BEAM.

Test T-24 is of interest because it had the lowest end-fixity efficiency of the 6-ft beam sections, with an end fixity of only 45.5 per cent. The remaining 6-ft sections were more than 85% efficient in end fixity. The explanation is found in the fact that the end stiffeners in this base were not properly designed, and the beam not only pulled loose at the weld from the end plate, but the end plate was badly warped during the test. The stiffeners were $\frac{9}{16}$ in. thick and 6 in. long. Had they been 9 in. long and 1 in. thick, as provided in the tentative Design Suggestions (1) and (2) (see heading, "Design of End Connection"), it is believed that much higher efficiency would have resulted.

Tests T-27 to T-29 are of three different lengths of 12-in. **I**-beam. The remarks concerning Tests T-19 to T-21 appear to apply in the case of these beams also.

SUMMARY AND CONCLUSIONS

Torsional Theory and the Torsion Constant.—The material presented in this paper may be outlined under eight divisions, as follows:

(1) The essential features of the general torsion problem are outlined;

(2) The application of Prandtl's membrane analogy is presented;

(3) An accurate and detailed method of evaluating the torsion constant of structural **H**-beams and **I**-beams is presented (this method is based on a known theoretical evaluation combined with factors determined by experiment from the membrane analogy);

(4) Formulas are proposed for the shearing stress in the flange and web due to pure torsion;

(5) The effect of shearing stress concentration in the fillets due to both torsion and bending is discussed;

(6) The problem of torsion with either one or both ends of the beam restrained is studied in detail and formulas for maximum shearing and longitudinal fiber stresses are given;

(7) Specifications are suggested for the design of a welded end connection which will give a high degree of end-fixity efficiency in torsion; and,

(8) Detailed design examples are presented to illustrate the application of the formulas to torsional design.

Test Results.—The purpose of the soap film tests was to determine, experimentally, the additive portion in the K-factor due to the added rigidity accruing from the juncture of the web and flange with the corresponding fillets. To this end, soap film experiments were made on fifty-seven differently proportioned sections both with parallel and sloping flanges. The application of the membrane analogy as developed by this investigation was restricted to volume measurement only. The method used was rapid, and, it is believed, gave results having an error considerably less than $\pm$ 1 per cent.

In computing the K-values of structural sections the experimentally determined part of K amounts, in the most extreme case, to 10% of the total K. Hence, an experimental error of $\pm$ 1% would give a possible error of only 0.1% in the total K-value.

Free-end torsion tests were made on seven different beams ranging in size from a 6-in. **H**-beam @ 20 lb per ft to a 12-in. **H**-beam @ 190 lb per ft. The heaviest beam had a torsion constant about two hundred times as great as the lightest beam.

The method of applying the torque to the ends of the beam provided a high degree of end freedom. By measuring the unit twist of the free-ended beams and obtaining the slope of the torque-twist diagram, the free-ended torsion tests provided (through Equation (5)) a definite check on the torsion constant as computed by the proposed method. Using G = 11 150 000 lb per sq in., which is theoretically correct for E = 29 000 000 lb per sq in. and μ = 0.30, the test results checked well with computed K-values. The maximum variation was 6.7% and the average for the seven tests was 2.26 per cent.

The yield point of the beams in both free-ended and fixed-ended tests was determined by a study of the torque-twist diagram and in some cases by a drop of the beam. In the free-ended tests the distribution of shearing stress across the flange was studied by measuring, with tensometers, the principal stress at an angle of 45° around the center section.

The fixed-ended tests provided a means of studying the additional rigidity and strength over the free-ended tests, due to fixing the ends of the beams. Twenty-two various lengths and sizes of beams were tested ranging from a 3-in. I-beam at 7.5 lb per ft, to a 10 by 12-in. beam at 62 lb per ft. The heaviest beam had a K-factor fifty-seven times greater than the least, and with ends fixed the most rigid fixed-end beam had a rigidity three hundred and ten times as great as the least rigid with ends fixed.

The distribution of longitudinal direct stresses in the extreme fibers of the flanges was studied by strain measurements taken along the outer edges of the flanges by tensometers. The total shearing stress distribution at the center section was studied by measurement of principal strains at an angle of 45 degrees.

The fixed-ended tests furnished information as to the proper design of welded end connections for high torsional rigidity.

The tensometers in both the free-ended and fixed-ended tests were of value in studying the relative distribution of stress. Some of the results are erratic. The warping of the section during twist made it difficult to obtain a steady set-up for the tensometers, but in most cases the test results checked fairly well with the formulas. In spite of variations in result and incomplete end fixity it is noted that every beam tested would have been amply strong if designed on a basis of working longitudinal stresses at the ends.

The present investigation has covered, accurately, the question of torsional rigidity and the evaluation of the torsion constant. The formulas for stresses which are proposed are not exact but will be satisfactory for practical design purposes.

Acknowledgment

The torsional investigation was conducted as a co-operative project, the McClintic-Marshall Corporation, subsidiary to the Bethlehem Steel Company, furnishing the steel beams and the material for constructing the test apparatus, and Lehigh University bearing all other expenses.

Through the courtesy of the Baldwin-Southwark Corporation, of Philadelphia, Pa., sixteen tensometers were loaned to the University (without charge), during these tests.

Special acknowledgment is due Jonathan Jones, M. Am. Soc. C. E., Chief Engineer; C. H. Mercer, M. Am. Soc. C. E., Consulting Engineer, and Sterling Johnston, M. Am. Soc. C. E., Engineer, all of the McClintic-Marshall Corporation, and V. E. Ellstrom, Manager of Sales Engineering, Bethlehem Steel Company, for their co-operation in the investigation.

The study was conducted as a regular research project of the Fritz Engineering Laboratory, Lehigh University. C. C. Keyser, Laboratory Assistant, and all others associated with the Laboratory contributed valuable help throughout. Professor J. B. Reynolds, of the Department of Mathematics, gave his assistance to the theoretical studies.

APPENDIX I

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APPENDIX II

NOTATION

The following symbols, adopted for use in this paper, are presented as a guide to discussers:

a = a torsional factor in fixed-ended beams.
b = length of a rectangular section.
d = depth; total depth of beam.
e = eccentricity.
f = a subscript denoting "flange."
h = distance between flange centroids.
i = $\frac{1}{2}(d - h)$.
m = major flange thickness.
n = breadth of a rectangular section; also, where defined, n = minor flange thickness.
p = pressure per unit area.
r = radius of a fillet (authors' duplicate); also, where defined, r = variable thickness of a section.
s = distance between stiffening plates.
t = thickness; as a subscript, t, denotes "due to bending."
u = a substitution factor in Equation (37); as a subscript, u, denotes "free ended."
w = web thickness; as a subscript, w, denotes "web."
y = deflection.
A = area; cross-section area; also, where specifically defined, A, B, C, and D are coefficients in a general equation; as a subscript, A, denotes "average."
B = a substitution factor in Equation (23) (see, also, Symbol A).
C = constant; torsion constant equivalent to K for a fixed-ended beam; $C_c = C$ at the center; C_A = average value of C (see, also, Symbol A).
D = diameter; diameter of an inscribed circle (see, also, Symbol A).
E = elasticity; modulus of elasticity in tension and compression.
F = torsion stress function.
G = shearing modulus of elasticity.
I = rectangular moment of inertia; I_x and I_y = I for a cross-section with respect to the X-axis and the Y-axis, respectively.
J = polar moment of inertia.
K = a torsion constant; K_f and K_w = values of K for flange and web, respectively.
L = length; as a subscript, L, denotes "large end."
M = bending moment.
Q = total shear over a cross-section.
S = total slope of a flange section $= \dfrac{m - n}{b}$; as a subscript, S, denotes "small end."
T = torque; torsional moment; T_a = torque required to twist a beam in a free-ended condition.

V = a factor depending on the $\frac{b}{n}$-ratio, but practically constant for $\frac{b}{n}$ greater than 3; V_L and V_S = value of V for large end and small end, respectively, of the flange.

α = a factor that depends on two ratios, $\frac{w}{m}$ and $\frac{r}{m}$.

ε = unit elongation.

θ = angle of twist, in radians per unit length; $\theta_c = \theta$ at the center.

μ = Poisson's ratio.

ρ = radius of curvature.

σ = unit stress; normal stress; σ_t = longitudinal fiber stress in a flange due to bending.

τ = unit shearing stress; $\tau_f = \tau$ for flange sections; $\tau_w = \tau$ for web sections.

ψ = total angle change between two cross-sections.

APPENDIX III

Tables

The tables presented herewith are introduced and adequately discussed in the text of this paper.

TABLE 1.—General Summary of All Tests

Test No.	Nominal size	Weight, in Pounds per Foot: Nominal	Weight, in Pounds per Foot: Actual	Length, in inches	Measured Dimensions, in Inches, Adjusted to the Actual Weight (See Fig. 6): Total depth, d	Length, b	Web thickness, w	Flange thickness: m	Flange thickness: n		Measured diameter, D, of inscribed circle, in inches	Torsion constant, K
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
T-4*	3-in. I-beam	7.5	7.4	3.2	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-5*	3-in. I-beam	7.5	7.4	6.0	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-6*	3-in. I-beam	7.5	7.4	9.0	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-7*	3-in. I-beam	7.5	7.4	12.1	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-8*	3-in. I-beam	7.5	7.4	18.1	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-9*	3-in. I-beam	7.5	7.4	23.9	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-10*	3-in. I-beam	7.5	7.4	39.0	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-11*	3-in. I-beam	7.5	7.4	53.9	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-12*	3-in. I-beam	7.5	7.4	5.5	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-13*	3-in. I-beam	7.5	7.4	8.8	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-14†	6 by 6-in.	20	19.6	72	6.06	6.01	0.257	.....	0.362	0.28	0.566	0.243
T-15*	6 by 6-in.	20	19.6	72	6.06	6.01	0.257	.....	0.362	0.28	0.566	0.243
T-16*	6 by 6-in.	20	20.9	72	6.08	6.06	0.303	.....	0.372	0.24	0.581	0.281
T-17*	6 by 6-in.	40.5	39.0	72	6.73	6.25	0.501	.....	0.699	0.35	0.928	1.738
T-18*	6 by 10-in.	40	38.1	72	6.20	9.91	0.431	.....	0.447	0.35	0.735	0.802
T-19*	8 by 8-in.	31	29.4	18	8.05	8.04	0.290	.....	0.401	0.41	0.676	0.463
T-20*	8 by 8-in.	31	29.4	36	8.05	8.04	0.290	.....	0.401	0.41	0.676	0.463
T-21*	8 by 8-in.	31	29.4	72	8.05	8.04	0.290	.....	0.401	0.41	0.676	0.463
T-22†	8 by 8-in.	31	29.4	72	8.05	8.04	0.290	.....	0.401	0.41	0.676	0.463
T-23*	8 by 8-in.	67	66.1	72	9.05	8.29	0.606	.....	0.907	0.42	1.208	4.912
T-24*	10 by 12-in.	62	60.9	72	10.05	11.92	0.390	0.659	0.546	0.45	0.919	2.101
T-25†	10 by 12-in.	62	60.9	72	10.05	11.92	0.390	0.659	0.546	0.45	0.919	2.101
T-26†	12-in. I-beam	31.8	31.3	72	12.06	4.97	0.350	0.705	0.345	0.42	0.882	0.833
T-27*	12-in. I-beam	55	53.6	18	12.07	5.74	0.730	0.900	0.449	0.54	1.244	3.318
T-28*	12-in. I-beam	55	53.6	36	12.07	5.74	0.730	0.900	0.449	0.54	1.244	3.318
T-29*	12-in. I-beam	55	53.6	72	12.07	5.74	0.730	0.900	0.449	0.54	1.244	3.318
T-30†	12-in. I-beam	55	53.6	72	12.07	5.74	0.730	0.900	0.449	0.54	1.244	3.318
T-31†	12 by 12-in.	190	186.7	72	14.39	12.56	1.069	.....	1.714	0.62	2.186	48.437
T-33†	12 by 12-in.	65	65.6	72	12.14	12.03	0.409	.....	0.604	0.58	0.953	2.227

* Fixed end. † Free end.

TABLE 2.—Physical Tests of Material in Test Beams

Samples; taken from Test Beam No.	Tensile Properties						Shearing Properties, in Pounds per Square Inch				Shearing modulus, G, in thousands of pounds per square inch
	Stresses, in Pounds per Square Inch			Modulus of elasticity E, in thousands of pounds per square inch	Percentage elongation in 2 inches	Percentage reduction in area	Slotted Plates		Round Bar Torsion Tests		
	Upper yield point	Lower yield point	At ultimate				Shearing yield point	Shearing ultimate	Shearing yield point	Apparent shearing ultimate	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
T-4, *T*-13	38 900	......	63 550	29 100	32.0	62.1	21 800	52 850	......	......	....
T-14, *T*-15	41 000	37 860	60 450	29 200	35.8	66.4	25 600	48 500	26 400	80 600	12 080
T-16	39 710	35 790	58 130	29 150	35.5	66.7	22 000	45 100	22 000	50 700	11 280
T-17	43 020	37 000	60 220	28 500	37.0	67.3	23 700	49 900	24 000	64 100	12 020
T-18	34 180	31 570	56 570	28 250	40.0	65.4	22 600	47 900	22 200	65 600	11 500
T-19, *T*-22	42 880	38 290	61 280	29 450	36.0	66.1	25 500	51 000	22 300	64 700	10 780
T-23	34 890	31 580	61 180	28 600	38.8	66.1	21 200	50 700	20 200	65 700	11 780
T-24, *T*-25	40 860	35 530	59 600	29 150	36 3	64 9	23 700	49 800	24 400	64 100	12 350
T-26	38 310	33 900	56 330	29 400	38.0	69.3	24 600	52 800	21 400	71 000	11 920
T-27, *T*-30	32 380	30 990	59 850	29 400	34.5	65.9	18 200	47 700	20 900 18 700	66 200 64 500	11 770 10 780
T-31	32 910	28 620	60 000	29 200	38.0	63.7	21 200	51 600	20 300	64 600	11 260
Average..	38 090	34 110	59 740	29 040	36.5	65.8	22 740	49 800	22 070	65 620	11 590

TABLE 3.—Tests on Free-Ended Beams

Test No.	Nominal size	Nominal weight, in pounds per foot	Torsion Constant, K			Yield point of beam, in inch-pounds	Shearing Stresses, in Pounds per Square Inch				
							Maximum, at the Yield Point				Yield point strength of material
							In the Flange		In the Web		
			From measured dimensions	By test	Percentage variation		By Equation (25)	By tensometers	By Equation (27)	By tensometers	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
T-14.......	6 by 6-in.....	20	0.243	0.243	0.0	15 900	30 400	29 700	22 400	19 700	26 000
T-22.......	8 by 8-in.....	31	0.463	0.451	−2.6	25 610	29 800	31 400	22 600	22 200	23 900
T-25.......	10 by 12-in...	62	2.101	1.960	−6.7	50 900	19 100	23 400	12 600	17 900	24 050
T-26.......	12-in I-beam..	31.8	0.833	0.804	−3.5	25 000	23 800	25 400	14 900	17 700	23 000
T-30.......	12-in. I-beam..	55	3.318	3.380	+1.9	64 000	20 700	20 200	17 100	18 500	19 000
T-31.......	12 by 12-in...	190	48.440	48.870	+0.9	480 000	19 300	21 400	12 400	12 600	20 750
T-33.......	12 by 12-in...	65	2.227	2.222	−0.2	76 400	26 600	29 300	......	......	......
Average..	....	....	......	......	2.26	......	24 240	25 830	17 000	18 100	22 780

TABLE 4.—TESTS OF FIXED-ENDED BEAMS

Test No.	Nominal size	Nominal weight, in pounds per foot	Length of beam, l, in inches	Moment of inertia, I_y, in inches^4 (from measured dimensions)	Torsion constant, K, from measured dimensions	Values of coefficient, $a = 0.806\, h \sqrt{\frac{I_y}{K}}$	Values of ratio, $\frac{l}{2a}$	TORSION CONSTANTS: C_e, computed by Equation (46)	TORSION CONSTANTS: C_A, Computed	TORSION CONSTANTS: C_A, By test
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
T-4	3-in. I-beam	7.5	3.2	0.57	0.0865	5.67	0.282	0.7998	0.943	0.787
T-5	3-in. I-beam	7.5	6.0	0.57	0.0865	5.67	0.529	0.4558	0.605	0.474
T-6	3-in. I-beam	7.5	9.0	0.57	0.0865	5.67	0.794	0.2823	0.394	0.330
T-7	3-in. I-beam	7.5	12.1	0.57	0.0865	5.67	1.067	0.1996	0.284	0.258
T-8	3-in. I-beam	7.5	18.1	0.57	0.0865	5.67	1.596	0.1341	0.190	0.174
T-9	3-in. I-beam	7.5	23.9	0.57	0.0865	5.67	2.108	0.1102	0.154	0.153
T-10	3-in. I-beam	7.5	39.0	0.57	0.0865	5.67	3.439	0.0913	0.120	0.116
T-11	3-in. I-beam	7.5	53.9	0.57	0.0865	5.67	4.753	0.0875	0.109	0.111
T-12	3-in. I-beam	7.5	5.5	0.57	0.0865	5.67	0.485	0.5004	0.654	0.534
T-13	3-in. I-beam	7.5	8.8	0.57	0.0865	5.67	0.776	0.2904	0.404	0.377
T-15	6 by 6-in.	20	72.0	13.1	0.2433	33.7	1.068	0.6186	0.905	0.777
T-16	6 by 6-in.	20	36.0*	13.8	0.2810	32.4	0.556	1.896	2.761	1.106
T-17	6 by 6-in.	40.5	72.0	28.5	1.7375	19.7	1.872	2.431	3.445	3.333
T-18	6 by 10-in.	40	72.0	72.5	0.8018	44.1	0.816	2.917	4.264	3.944
T-19	8 by 8-in.	31	18.0	34.8	0.4631	53.5	0.168	20.927	27.55	27.90
T-20	8 by 8-in.	31	36.0	34.8	0.4631	53.5	0.336	7.496	10.75	7.43
T-21	8 by 8-in.	31	72.0	34.8	0.4631	53.5	0.673	2.346	3.45	3.30
T-23	8 by 8-in.	67	72.0	86.2	4.9122	27.4	1.314	9.479	13.67	12.00
T-24	10 by 12-in.	62	72.0	162.7	2.101	67.0	0.537	15.104	22.03	10.02
T-27	12-in. I-beam	55	18.0	18.9	3.318	21.8	0.413	32.059	44.30	34.50
T-28	12-in. I-beam	55	36.0	18.9	3.318	21.8	0.826	11.636	16.89	11.23
T-29	12-in. I-beam	55	72.0	18.9	3.318	21.8	1.651	5.1675	7.39	6.58

Test No.	End fixity (percentages)	Yield point of beam, by test in inch-pounds	UNIT STRESSES, IN POUNDS PER SQUARE INCH: Maximum direct stress at end, by Equation (40)	Direct stress near end by tensometers	Maximum Shearing Stress at the Yield Point: In the Flange: By Equation (47)	In the Flange: By tensometers	In the Web: By Equation (48)	In the Web: By tensometers
(1)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
T-4	83.5	26 000	58 700	....	30 500	....	....	....
T-5	78.3	16 000	70 500	....	24 500	....	....	....
T-6	83.8	10 000	60 300	....	20 360	....	....	....
T-7	90.8	8 500	61 100	....	22 000	....	....	....
T-8	91.6	6 500	54 600	....	22 800	....	....	....
T-9	99.4	6 160	54 500	....	25 400	....	....	....
T-10	96.8	5 500	50 100	....	26 400	....	....	....
T-11	101.8	5 400	49 300	....	27 000	....	....	....
T-12	84.2	8 000†	32 900†	....	11 600†	....	....	....
T-13	93.4	6 000†	35 600†	....	12 000†	....	....	....
T-15	85.8	33 980	72 600	66 100†	28 000	29 600	18 700	13 500
T-16	40.1	35 000	44 100	61 500†	11 100	23 200	....	....
T-17	96.8	79 000	54 000	40 000	27 800	29 000	19 700	16 400
T-18	92.5	69 000	48 700	30 300	17 000	16 700	....	....
T-19	101.0	151 600	40 700	49 000	13 000	25 800	....	....
T-20	69.1	115 000	60 000	56 700	14 900	19 100	....	....
T-21	95.7	65 000	61 600	56 300†	18 100	19 500	....	....
T-23	87.8	194 000	53 300	37 400	24 000	24 800	15 000	14 500
T-24	45.5	223 000	57 000	48 100	16 200	21 600	7 800	13 450
T-27	77.9	310 000	71 000	29 400	17 700	30 000	8 600	5 920
T-28	66.6	162 000	64 300	68 500	17 900	24 400	12 400	13 800
T-29	89.0	105 000	57 100	65 800	22 700	20 200	18 100	23 740

* 36 in. = one-half total length on account of additional stiffener in middle. † Not yield point.

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DISCUSSIONS

STRUCTURAL BEAMS IN TORSION

Discussion

By Messrs. H. M. Westergaard and R. D. Mindlin, and Joseph B. Reynolds

H. M. Westergaard,[12] M. Am. Soc. C. E., and R. D. Mindlin,[13] Jun. Am. Soc. C. E. (by letter).[13a]—In Fig. 9 the authors show curves which represent different formulas for the concentration of torsional shearing stresses at fillets. The curve which is identified by the names of the writers was derived as a result of correspondence with the authors, and for this reason its derivation is given herein.

The process of approximate analysis is closely related to those used by A. and L. Föppl[14] and by S. Timoshenko,[15] but differs in the following respects:

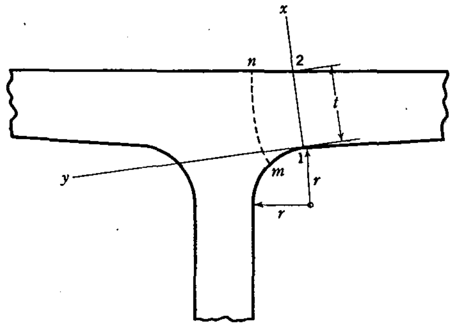

Fig. 27.—I-Beam in Torsion.

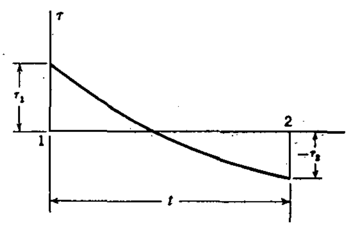

Fig. 28.—Shearing Stresses at Section 1-2, in Fig. 27.

First, consideration of the influence of the straight edge of the cross-section opposite the fillet led to a slightly greater concentration factor near the beginning of the fillet, at Point 1 in Fig. 27, than that obtained by Timoshenko's

Note.—The paper by Inge Lyse, M. Am. Soc. C. E., and Bruce G. Johnston, Jun. Am. Soc. C. E., is published in this number of *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

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[13a] Received by the Secretary March 1, 1935.

[14] "Drang und Zwang," by A. and L. Föppl, Vol. 2, Second Edition, 1928, p. 73.

[15] "Theory of Elasticity," by S. Timoshenko, 1934, p. 259.

formula; and, second, the writers had before them the detailed data obtained by P. A. Cushman[16] in tests with soap films. These data show clearly that the maximum shearing stress occurs at a point such as m in Fig. 27, and that there is a notable increase of stress from Point 1 to Point m. Special consideration was given to this increase.

Let τ_0 denote the shearing stress that would exist at Point 1 in Fig. 27 if the edge of the cross-section at that point were straightened out by moving the beginning of the fillet toward the left, when, at the same time, the thickness of the flange, t, the torsion factor, K, and the total twisting moment, T, are left unchanged. The stress, τ_0 is defined by the formula:

$$\tau_0 = \frac{T\,t}{K} \qquad (57)$$

Let τ_1 and τ_m denote the actual shearing stresses at Points 1 and m, τ_m being the maximum stress at the fillet. Then $\frac{\tau_1}{\tau_0}$ is the concentration factor at Point 1 and $\frac{\tau_m}{\tau_0}$ is the desired concentration factor for the fillet.

Prandtl's soap film analogy, which the authors have used advantageously, furnishes the key to the solution. It is noted that the soap film is stretched over an opening shaped like the cross-section and is inflated a small amount by an excess air pressure on one side. The shearing stresses on the original cross-section follow the contour lines on the film, the edge being one of the contour lines. Furthermore, the shearing stresses are proportional to the slopes of the film, or proportional to the density of the contour lines.

The contour lines at Section 1–2 in Fig. 27 must be approximately perpendicular to that section. Accordingly, the shearing stresses, τ, at that section are approximately in the direction of y. Since the slope of the film is $c\tau$, c being a constant, the curvature of the film at Section 1–2 in the direction of x becomes $c\,\frac{d\tau}{dx}$. The curvature of the film in the direction of y at Section 1–2 is accounted for by the curving of the contour lines. If the radius of curvature of the contour lines at a particular point is R, then the curvature of the film in the direction of y at the same point becomes $c\,\frac{\tau}{R}$. The curvature of the surface is the sum of the curvatures in the directions of x and y; that is, $c\left(\frac{d\tau}{dx} + \frac{\tau}{R}\right)$.

The equilibrium of the film requires that this combined curvature be constant. Since $R = r$ at Point 1 and $R = \infty$ at Point 2, it follows that,

$$\left[\frac{d\tau}{dx}\right]_1 + \frac{\tau_1}{r} = \left[\frac{d\tau}{dx}\right]_2 \qquad (58)$$

[16] "Shearing Stresses in Torsion and Bending by Membrane Analogy," by P. A. Cushman, Doctoral Dissertation, Univ. of Michigan, 1932.

Since Points 1 and 2 are on the same contour line, it is also required that,

$$\int_0^t \tau \, dx = 0 \quad \text{.......................} (59)$$

The diagram of the shearing stresses at Section 1-2 must be shaped about as shown in Fig. 28. Equation (60) has been constructed so that it satisfies this general requirement of shape as well as the specific requirements in Equations (58) and (59), and, therefore, it may be assumed to represent the shearing stress approximately:

$$\tau = \tau_1 \left[1 - \left(2 + \frac{t}{3r} \right) \frac{x}{t} + \frac{x^2}{2\,tr} \right] \quad \text{.............} (60)$$

The constant curvature of the film may be computed as:

$$c \left[\frac{d\tau}{dx} \right]_2 = - \frac{2\,c\,\tau_1}{t} \left(1 - \frac{t}{3r} \right) \quad \text{................} (61)$$

If the fillet were some distance away, this curvature would remain the same, τ_1 would be replaced by τ_0, and the term containing r would disappear. Consequently,

$$\tau_0 = \tau_1 \left(1 - \frac{t}{3r} \right) \quad \text{.....................} (62)$$

which gives the concentration factor at Point 1,

$$\frac{\tau_1}{\tau_0} = \frac{1}{1 - \dfrac{t}{3r}} \quad \text{........................} (63)$$

The important indications of Equation (63) are preserved when the following simpler formula is substituted:

$$\frac{\tau_1}{\tau_0} = 1 + \frac{t}{3r} \quad \text{........................} (64)$$

As was mentioned, the maximum stress, τ_m, occurs at a point such as m in Fig. 27. Equation (64) can be used to obtain an estimate of this stress by making the following replacements: t is replaced by a distance, t_m, measured along a curved section as drawn from m to n; and τ_0 is replaced by a stress corresponding to straight edges and a thickness, t_m: that is, according to Equation (57), by $\tau_0 \dfrac{t_m}{t}$. Thus, the concentration factor becomes,

$$\frac{\tau_m}{\tau_0} = \frac{t_m}{t} \left(1 + \frac{t_m}{3r} \right) \quad \text{.....................} (65)$$

A reasonable estimate of t_m is,

$$t_m = t + 0.3\,r \quad \text{.......................} (66)$$

This value, substituted in Equation (65), leads directly to the formula represented graphically by the authors,

$$\frac{\tau_m}{\tau_0} = 1.2 + \frac{1}{3}\left(\frac{t}{r} + \frac{r}{t}\right) \qquad (67)$$

JOSEPH B. REYNOLDS,[17] ESQ. (by letter).[17a]—The problem of the twisting of I-beams has been studied thoroughly by the authors for special cases. It is the purpose of this discussion to compare the authors' results with a more general theory for the twisting of I-beams under differing types of loading. Only the effect of twisting is considered. The assumptions and the notation are the same as those used in the paper.

Consider the requirements for equilibrium at a section of the beam at a distance, x, from the fixed end as shown in Fig. 29. The inner moments at

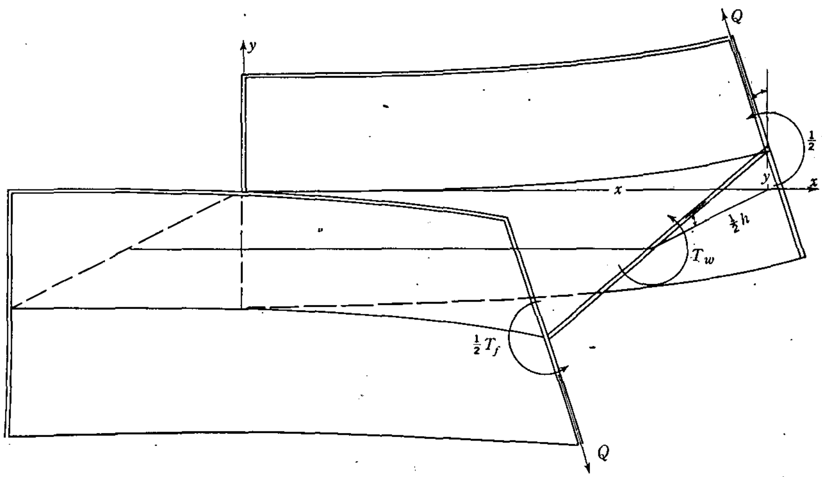

FIG. 29.

this section, together with the moment of the shears, must balance the outer moment applied to the beam to the right of the section. This outer twisting moment, T_x, will vary in general with the position of the section; that is, with x. For equilibrium at this section one must have:

$$T_f + T_w + Q\,h = T_x \qquad (68)$$

in which T_f, T_w, and T_x indicate twisting moments of flanges, web, and total of section, respectively.

In terms of the co-ordinates of points on the neutral axis, $y = f(x)$, the following approximate relation may be written:

$$\frac{1}{2}\;h\;d\theta = dy \qquad (69)$$

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[17a] Received by the Secretary March 16, 1935.

and the twist per unit length is $\frac{d\theta}{dx}$, hence:

$$T_f + T_w = KG \frac{d\theta}{dx} = \frac{2\,K\,G}{h} \frac{dy}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (70)$$

From the relation between shear and bending moment:

$$\frac{1}{2} E_y I \frac{d^3 y}{dx^3} = - Q \quad \ldots\ldots\ldots\ldots\ldots\ldots (71)$$

By means of Equations (70) and (71) and the value of a, Equation (68) can be written:

$$a^2 \frac{d^3 y}{dx^3} - \frac{dy}{dx} + X = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (72)$$

in which, $X = \frac{h\,T_x}{2\,K\,G}$.

The differential in Equation (72), has the general solution:

$$y = A \sinh \frac{x}{a} + B \cosh \frac{x}{a} + C + y_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (73)$$

in which y_1 is the particular solution satisfying Equation (72) and the values of A, B, and C are determined by the boundary conditions for the beam. Equation (73) is the same as the authors' Equation (34) for the case considered. The value of y_1 is:

$$y_1 = \int X\,dx - \frac{1}{2}\,a\,e^{\frac{x}{a}} \int e^{\frac{-x}{a}}\,X\,dx - \frac{1}{2}\,a\,e^{\frac{-x}{a}} \int e^{\frac{x}{a}}\,X\,dx \quad \ldots (74)$$

For the usual types of loading, X can be written in the form:

$$X = X_0 + X_1\,x + X_2\,x^2 + \ldots + X_n\,x^n \quad \ldots\ldots\ldots (75)$$

in which, X_0, X_1, X_2, ... X_n, are constants. When Equation (75) holds successive integration by parts shows that Equation (74) may be written in the form:

$$y_1 = \int X\,dx + a^3 \left(\frac{dX}{dx} + a^2 \frac{d^3 X}{dx^3} + a^4 \frac{d^5 X}{dx^5} + \ldots \right) \quad \ldots (76)$$

If n is even in Equation (75) the last term in Equation (76) is $\frac{d^{n-1} X}{dx^{n-1}}$; if n is odd, the last term is $\frac{d^n X}{dx^n}$. With the value of y_1 thus determined one may, by Equation (73) and the relation, $M = \frac{1}{2}\,EI\,\frac{d^2 y}{dx^2}$, write:

$$M = \frac{2\,K\,G}{h^2} \left[A \sinh \frac{x}{a} + B \cosh \frac{x}{a} + a^2 \left(\frac{dX}{dx} + a^3 \frac{d^3 X}{dx^3} + a^5 \frac{d^5 X}{dx^5}, \ldots \right) \right] . (77)$$

and, since the shear is given by $Q = \frac{1}{2} EI \frac{d^3 y}{dx^3}$:

$$Q = \frac{2\,K\,G}{a\,h^2}\left(A \cosh \frac{x}{a} + B \sinh \frac{x}{a} + a^3 \frac{d^2 X}{dx^2} + a^6 \frac{d^4 X}{dx^4} + \ldots\right) \quad (78)$$

The longitudinal stresses along the outer fibers of the flange will be given by $\sigma = \frac{M\,b}{I_y}$, or by:

$$\sigma = \frac{Eb}{2\,a^2}\left(A \sinh \frac{x}{a} + B \cosh \frac{x}{a} + a^2 \frac{d\,X}{dx} + a^5 \frac{d^3 X}{dx^3} + a^7 \frac{d^5 X}{dx^5} + \ldots\right) \quad (79)$$

and the lateral shearing stresses in the flanges by $\tau = \frac{Qb^2}{4\,I_y}$, or by:

$$\tau = \frac{Eb^2}{8\,a^3}\left(A \cosh \frac{x}{a} + B \sinh \frac{x}{a} + a^3 \frac{d^2 X}{dx^2} + a^6 \frac{d^4 X}{dx^4} + \ldots\right) \quad (80)$$

The angle through which the beam twists comes from the relation, $\theta = \frac{2\,y}{h}$, and has the value:

$$\theta = \frac{2\,A}{h} \sinh \frac{x}{a} + \frac{2\,B}{h} \cosh \frac{x}{a} + \frac{2\,C}{h} + \frac{2}{h}\int X\,dx + \frac{2\,a^3}{h} \times \frac{dX}{dx}$$
$$+ \frac{2\,a^5}{h} \times \frac{d^3 X}{dx^3} + \frac{2\,a^7}{h} \times \frac{d^5 X}{dx^5} + \ldots \quad (81)$$

In this manner, general values are derived for the principal variables of interest in the case of twisted beams. The maximum values of these quantities will occur for differing values of x, depending upon the conditions surrounding the strained beam. Example (a) demonstrates that Equations (73) to (81) reduce to those given by the authors when the proper limitations are applied.

Example (a).—*Beam Twisted by Constant Torque, T, Applied at Its Ends, with Both Ends Restrained.*—In this case, $T_x = T$, a constant, and, therefore, $X = \frac{h\,T}{2\,K\,G}$. The constants A, B, and C are determined by the requirements that for $x = 0$, $y = 0$, and $\frac{dy}{dx} = 0$. Furthermore, for $x = \frac{l}{2}$, $\frac{d^2 y}{dx^2} = 0$. Since the derivations of X are all zero, Equation (76) yields $y_1 = \frac{h\,T\,x}{2\,K\,G}$ and, by Equation (73):

$$y = \frac{T\,h\,a}{2\,K\,G}\left(\cosh \frac{x}{a} \tanh \frac{l}{2\,a} - \sinh \frac{x}{a} + \frac{x}{a} - \tanh \frac{l}{2\,a}\right) \quad (82)$$

which is the same as that presented by the authors in Equation (35). The maximum displacement, y_m, occurs where $x = l$ and has the value given in the authors' Equation (36).

Similarly, Equation (77) reduces to the authors' Equation (37) for the moment in each flange, which has (where $u = \frac{1}{2}\frac{l}{a}$, or $x = 0$), a maximum value:

$$M_m = \frac{T\,a}{h}\tanh\frac{l}{2\,a} \qquad (83)$$

Equation (78) reduces to Equation (38) and Equation (79) becomes Equation (39).

By use of Equation (80):

$$\tau = \frac{T\,b^2}{4\,I_y}\,\frac{\cosh u}{\cosh\frac{l}{2\,a}} \qquad (84a)$$

and,

$$\tau_{max} = \frac{T\,b^2}{4\,I_y} \qquad (84b)$$

which corresponds to the authors' Equation (42), and by Equation (81):

$$\theta = \frac{T\,a}{G\,K}\left(\cosh\frac{x}{a}\tanh\frac{l}{2\,a} - \sinh\frac{x}{a} + \frac{x}{a} - \tanh\frac{l}{2\,a}\right) \qquad (85)$$

from which,

$$\theta_{max} = \frac{T\,a}{G\,K}\left(\frac{l}{a} - 2\tanh\frac{l}{2\,a}\right) \qquad (86)$$

Example (b).—Beam with One End Fixed, the Other Free, Under Constant Torque, T.—In this case, $X = \frac{h\,T}{2\,K\,G}$ as in Example (*a*). The conditions determining *A*, *B*, and *C* are that for $x = 0$, $y = 0$, and $\frac{dy}{dx} = 0$, and for $x = l$, $\frac{d^2 y}{dx^2} = 0$. The value of the deflection obtained is:

$$y = \frac{T\,h\,a}{2\,K\,G}\left(\cosh\frac{x}{a}\tanh\frac{l}{a} - \sinh\frac{x}{a} + \frac{x}{a} - \tanh\frac{l}{a}\right) \qquad (87)$$

From this all the other principal variables are readily found as in Example (*a*).

Example (c).—Beam with Both Ends Fixed and a Uniform Eccentric Load Along Its Length Producing an External Moment, T.—In this case, $T_x = T\left(1 - \frac{2\,x}{l}\right)$, and, $X = \frac{h\,T}{2\,K\,G}\left(1 - \frac{2\,x}{l}\right)$ giving,

$$y_1 = \frac{h\,T}{2\,K\,G}\left(x - \frac{x^2}{l}\right) - \frac{a^3\,h\,T}{K\,G\,l} \qquad (88)$$

The values of A, B, and C are determined by the requirements that for $x = 0$, $y = 0$, and $\frac{dy}{dx} = 0$, and for $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$. The deflection found is:

$$y = \frac{T\,h\,a}{2\,K\,G}\left(\frac{\cosh\,u}{\sinh\,\frac{l}{2\,a}} - \coth\,\frac{l}{2\,a} + \frac{x}{a} - \frac{x^2}{a\,l}\right) \qquad (89)$$

Example (d).—Beam with One End Fixed and the Other Free, Uniform Eccentric Load Along Its Length.—Here, $Tx = T\left(1 - \frac{x}{l}\right)$ and,

$$y_1 = \frac{h\,T}{2\,K\,G}\left(x - \frac{x^2}{2\,l}\right) - \frac{T\,h\,a^3}{2KGl}$$

The constants, A, B, and C, are determined by the requirements that for $x = 0$, $y = 0$, and $\frac{dy}{dx} = 0$, and for $x = l$, $\frac{d^2y}{dx^2} = 0$. The deflection proves to be:

$$y = \frac{T\,h\,a}{2\,K\,G}\left[\frac{\sinh\left(\frac{l}{a} - \frac{x}{a}\right)}{\cosh\,\frac{l}{a}} - \tanh\frac{l}{a} + \frac{a}{l}\,\mathrm{sech}\,\frac{l}{a}\left(\cosh\frac{x}{a} - 1\right) + \frac{x}{a} - \frac{x^2}{2\,a\,l}\right] \qquad (90)$$

It is thus shown that the formulas developed and substantiated by test results in the authors' investigation, are identical with those obtained by a general mathematical treatment of the subject of torsional resistance.

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